Solutions Manual for

Probability and Statistical Inference

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CHAPTER 1

1.2.1 **(i)** The outcomes may be represented as $(H, x), x = 1, \ldots, 6$ and (T, x, y), $x, y = 1, \ldots, 6$. One has to decide whether outcomes such as $(T, 1, 3)$ and $(T, 3, 1)$ should be regarded as different. If outcomes (T, x, y) and (T, x, y) are considered the same, the total number of points in S will be $6+21 = 27$. Otherwise, the number of points in S will be $6+36 = 42$. With the second choice, all points with 3 coordinates are equally likely. However, with the first choice, the point such as $(T, 1, 3)$ will be twice as likely as $(T, 1, 1)$. **(ii)** TT, HTT, THT, THHT, HTHT, HHTT, HHHH, THHH, HTHH, HHTH, HHHT.

1.2.2 **(i)** Assuming that the four persons are ordered (e.g., alphabetically), we can just write four floor numbers at which the corresponding persons leave the elevator. For instance, 3423 means that Alice and Diana leave at third floor, Bob at fourth and Carl at second floor (another notation may be $C|AD|B$, $ABD||C$ for 2242, etc.) **(ii)** 2222 2233 2244 2322 2333 2344 2422 2433 2444 3222 3233 3244 3322 3333 3344 3422 3433 3444 4222 4233 4244 4322 4333 4344 4422 4433 4444. **(iii)** The nine outcomes are : 2233, 2333, 2433, 3233, 3333, 3433, 4233, 4333, 4433, or $AB|CD||$, $A|BCD||$, $A|CD|B$, $B|ACD||$, $||ABCD||$, $||ACD|B$, $B|CD|A$, $||BCD|A$, $||CD|AB$.

1.2.3 The event in question occurs when chip "3" is drawn, one of the other chips has number less than 3, and the other has number greater than 3: 134, 234, 135, 235.

1.2.4 The event in question consists of infinitely many outcomes. One may simplify the description considering sums rather than individual results. The event "win with a point of 5" consists of outcomes of the form $55, 5o5, 5oo5, 5ooo5, \ldots$ where each o represents a result of the toss of pair of dice which is neither 5 nor 7, so that each o may be 2, 3, 4, 6, 8, 9, 10, 11, or 12.

1.2.5 **(i)** An outcome is specified by two numbers: the number x of balls in the first urn, and y, the number of balls in the second urn. Then $6 - x - y$ is the number of balls in the third urn. The idea is to propose some systematic way of listing pairs (x, y). For instance, start with largest possible x, and then largest possible y, etc. Thus we have configurations (the third digit gives the number of balls in the third urn): 600, 510, 501, 420, 411, 402, 330, 321, 312, 303, 240, 231, 222, 213, 204, 150, 141, 132, 123, 114, 105, 060, 051, 042, 033, 024, 015, 006. **(ii)** If the urns are indistinguishable, then the number of allocations is the same as the number of ways 6 can be expressed as a sum of three nonnegative integers: 600, 510, 420, 411, 330, 321, 222. **(iii)** Now the problem can be reduced to solving twice the problem (i) - separately for 2 white and 4 red balls. For 2 white balls we have 6 allocations: 200, 110, 101, 020, 011, 002. For 4 red balls we have 15 allocations: 400, 310, 301, 220, 211, 202, 130, 121, 112, 103, 141, 031, 022, 013, 004. Each allocation of 2 white and 4 red balls is obtained as a combination of two allocations, such as 110 (white) and 040 (red). Since any two can be combined, we obtain altogether $6 \times 15 = 90$ allocations.

1.2.6 Let x and y denote the times of arrival of John and Mary. Counting the time in minutes, starting from 5 p.m., the sample space is the square $0 \le x \le 60, 0 \le y \le 60$. **(i)** $0 \le x \le y \le 60$. **(ii)** $0 \le x \le y \le \min(60, x + 20)$ and $0 \le y \le x \le \min(60, y + 5)$. **(iii)** $0 \le y \le x \le 60$ or $x + 20 < y \le 60$. **(iv)** $y + 5 < x \le 60$. **(v)** Not precise enough to express as a set. **(vi)** $y + 5 \le x \le y + 15$. **(vii)** $0 \le x \le 15, y = 15$. **(viii)** Not precise enough to express as a set. Events in (i)-(iv) and (vi)-(vii) can be expressed graphically as subsets of a sample space (the square $0 \le x \le 60, 0 \le y \le 60$).

1.2.7 **(i)** THT. **(ii)** Events that specify the outcome of a particular toss cannot be expressed as subsets of $S^{'}$, since outcomes in $S^{'}$ do not convey information about the results of specific tosses. Such events are A_2, A_4, and A_6. We have $A_1 = \{3\}, A_3 = \{0, 1\}, A_5 = S^{'}$. **(iii)** HHH, TTT, HTT, THH. **(iv)** B_2, B_5. $B_1 = \{(1, 1), (0, 0)\}, B_3 = \{(1, 1)\}$, and $B_4 = B_3^c = \{(1, 1)\}^c$ are subsets of $S^{''}$.

1.2.8 **(i)** A_2, A_3, A_4, A_5. To show that these events are not subsets of $S^{'}$, it suffices to show for each event A_i above, two outcomes which have the same sums, one of them being in A_i, and the other not. Such outcomes are $(2, 2)$ and $(3, 1)$ for A_2; $(3, 5)$ and $(4, 4)$ for A_3; $(1, 2)$ and $(2, 1)$ for A_4; $(3, 4)$ and $(1, 6)$ for A_5. For the remaining sets, $A_1 = \{3, 5, 7, 9, 11\}, A_6 = \{12\}$. **(ii)** B_1 is not a subset of $S^{''}$ since $(1, 2)$ and $(2, 4)$ do not have the same absolute difference. The others are: $B_2 = \{|i - j| = 0\}, B_3 = \{|i - j| = 5\}, B_4 = \{|i - j| \text{ is odd }\}, B_5 = \{|i - j| \ge 1\}$.

1.2.9 Yes. The difference is that if the answer is "yes" and the interviewer manages to learn somehow that the respondent was not born in April, he knows that the respondent answered "yes" to the Q—question. Thus, in some cases confidentiality can be violated.

1.3.1 **(i)** False. Indeed, if A and B^c are disjoint, then $A \subset B$. Since $A \ne B$, there is a point, say x, which is in B but not in A. Then x belongs to both B and A^c, which means that A^c and B are not disjoint (a Venn diagram will help here). **(ii)** False. The asserted property holds only when $A \cup B = S$, in which case $A^c = B$ and $B^c = A$. **(iii)** False. Take A, B disjoint and $C = A$. **(iv)** True, make a Venn diagram. Formally $A \subset C$ implies $C^c \subset A^c$. Similarly, $C^c \subset B^c$, so that $C^c \subset A^c \cap B^c$. **(v)** True, subsets of disjoint sets must be disjoint. **(vi)** True.

1.3.2 (ii), (v), (vi), (viii), (x) are true.

1.3.3 **(i)** $X = A$. **(ii)** $X = \emptyset$. **(iii)** $X = A^c$. **(iv)** $X = B \div A$.

1.3.4 The number of students who take all three classes is $x = 2$.

1.3.5 $D_7 = E_1, D_2 = E_2, D_6 = E_3, D_8 = E_9, D_1 = E_{11}, D_9 = E_{10}, D_1 = E_6, D_5 = E_5 = E_4, D_{10} = E_7, D_3 = E_8$.

1.3.6 **(i)** 1. **(ii)** 3. **(iii)** 0. **(iv)** 0. **(v)** 4. **(vi)** 2.

1.3.7 **(i)** $x = 0, y = 4$. **(ii)** Either $x = 0$ or $y \geq 3$. **(iii)** $x = 0$, no inference about y possible. **(iv)** $x \leq 4, y \geq 1$.

1.3.8 $A^{nc} = A$ if n is even and $A^{nc} = A^c$ if n is odd. Consequently, $A^{mc} \cap A^{nc} = A$ if m, n are both even, $A^{mc} \cap A^{nc} = A^c$ if m, n are both odd and $A^{mc} \cap A^{nc} = \emptyset$ if m is odd and n is even or vice versa. For the union, $A^{mc} \cup A^{nc}$ equals A, A^c or S in the three cases above.

1.4.1 $\lim A_n = \emptyset$. Indeed, sequence A_n is monotone decreasing, so that $\lim A_n = \bigcap_{n=1}^{\infty} A_n$. The fact that B_k's form a partition implies $\bigcap_n A_n = \emptyset$.

1.4.2 **(i)** (b) is the only true statement. **(ii)** $C_1 = D_7; C_2 = D_1; C_3 = D_6 = D_{10}; C_4 = D_2; C_5 = D_5; C_6 = D_8; C_7 = D_3; C_8 = D_{11}; C_9 = D_4 = D_9$;

1.4.3 **(i)** There are maximally 2^n sets formed as intersections of all A_i's or their complements, i.e., sets such as $A_1 \cap A_2^c \cap A_3^c \cap \ldots \cap A_n$ (called *atoms*). Each set in the field containing $A_1, \ldots, A_n$ is a union of some atoms, so that the total possible number of sets in the field is 2^{2^n}. **(ii)** If $A_{n-1} \subset A_n$, then the sets A_{n-1} and A_n, or their complements, have only three (and not four) nonempty intersections, namely $A_{n-1} = A_{n-1} \cap A_n, A_{n-1}^c \cap A_n$ and $A_n^c = A_{n-1}^c \cap A_n^c$. Then the maximal possible number of intersections of atoms is $2^{n-2} \times 3$, and the number of sets in the field is $2^{3 \times 2^{n-1}}$. **(iii)** Now there are $n+1$ possible atoms, namely $A_1, A_2 \cap A_1^c, A_3 \cap A_2^c, \ldots, A_n \cap A_{n-1}^c$ and A_n^c. The number of sets in the field equals therefore 2^{n+1}. **(iv)** If $A_1 = \cdots = A_n = \emptyset$ then the field consists of two sets, $\emptyset$ and S. **(v)** Answers are the same: a field generated by finitely many sets is also a σ-field.

1.4.4 **(i)** Let $\beta_n = \inf_{k \geq n} \alpha_k$. Then $\beta_n \uparrow \beta = \liminf \alpha_n$. If $\beta_n < \beta$ for every n, then $\liminf A_n = I(\beta) = \{x : 1 - \beta < x < 1 + \beta\}$. If $\beta_n = \beta$ starting from some n, then $\liminf A_n = \overline{I(\beta)} = \{x : 1 - \beta \leq x \leq 1 + \beta\}$. As regards $\limsup A_n$, the situation is slightly different. Let $\gamma_n = \sup_{k \geq n} \alpha_k$, so that $\gamma_n \downarrow \gamma = \limsup \alpha_n$. If $\gamma_n > \gamma$ for every n, then $\limsup A_n = \overline{I(\gamma)} = \{x : 1 - \gamma \leq x \leq 1 + \gamma\}$. If $\gamma_n = \gamma$ starting from some n, two cases are possible: 1. There exists a subsequence $\{\alpha_{n_k}\}$ such that $\alpha_{n_k} = \gamma, k = 1, 2, \ldots$. In this case, $\limsup A_n = \overline{I(\gamma)} = \{x : 1 - \gamma \leq x \leq 1 + \gamma\}$. 2. Otherwise (i.e., when the sequence α_n satisfies the condition $\alpha_n < \gamma$ for all n sufficiently large, and only contains a subsequence $\{\alpha_{n_k}\}$ convergent monotonically to γ), then $\limsup A_n = I(\gamma) = \{x : 1 - \gamma < x < 1 + \gamma\}$. **(ii)** The limit $\lim A_n$ exists under the following conditions: 1. The sequence $\{\alpha_n\}$ becomes ultimately monotone decreasing, i.e., $\alpha_N \geq \alpha_{N+1} \geq \cdots$ for some N. Then $\alpha = \lim \alpha_n$ exists, and $\lim A_n = \overline{I(\alpha)} = \{x : 1 - \alpha \leq x \leq 1 + \alpha\}$. 2.The sequence $\{\alpha_n\}$ becomes ultimately monotone increasing, i.e., $\alpha_N \leq \alpha_{N+1} \leq \ldots$

for some N. Then $\alpha = \lim \alpha_n$ exists. If $\alpha_n < \alpha$ for all $n \geq N$, then $\lim A_n = I(\alpha) = \{x : 1 - \alpha < x < 1 + \alpha\}$. If $\alpha_n = \alpha$, starting with some N, then $\overline{I(\alpha)} = \{x : 1 - \alpha \leq x \leq 1 + \alpha\}$. **(iii)** The answers are the same as for (i) and (ii): the limits of sequences for sets B_n do not depend on closure properties of $I(\alpha_n)$ and $J(\alpha_n)$, and only on behavior of the sequence $\{\alpha_n\}$.

1.4.5 For A_1 we have $\lim f_n(A_1)/n = 1/2$. Similarly, $\lim f_n(A_2)/n = 1/2$, so that $A_1 \in \mathcal{A}$ and $A_2 \in \mathcal{A}$. Now, $A_1 \cap A_2$ consists of all odd integers between 2^{2n} and 2^{2n+1} for $n = 1, 2, \ldots$ that is of groups 5, 7; 17, 19, 21, 23, 25, 27, 29; 65, 67, ..., 127;.... Consequently, $f_{2^{2k-1}}(A_1 \cap A_2) = f_{2^{2k}}(A_1 \cap A_2) = 2 + 2^3 + \cdots + 2^{2k-3} = (2/3)(2^{2(k-1)} - 1)$, and

$$\begin{aligned} \frac{f_{2^{2k-1}}(A_1 \cap A_2)}{2^{2k-1}} &= \frac{2}{3} \times \frac{2^{2(k-1)} - 1}{2^{2k-1}} = \frac{2}{3}\left(\frac{1}{2} - \frac{1}{2^{2k-1}}\right) \to \frac{1}{3} \\ \frac{f_{2^{2k}}(A_1 \cap A_2)}{2^{2k}} &= \frac{2}{3} \times \frac{2^{2(k-1)} - 1}{2^{2k}} = \frac{2}{3}\left(\frac{1}{4} - \frac{1}{2^{2k}}\right) \to \frac{1}{6}. \end{aligned}$$

Consequently, $\lim f_n(A_1 \cap A_2)/n$ does not exist, hence $A_1 \cap A_2 \notin \mathcal{A}$.

1.4.6 The union and intersection of overlapping intervals of the form (a, b), $[a, b)$, $(a, b]$ or $[a, b]$ is again an interval of one of the four forms. If we allow infinite intervals, then a complement of an interval of the above form is again such an interval, or the union of two such intervals. It follows that class of finite unions of such intervals is closed under complementation, union and intersection, i.e., it is a field.

CHAPTER 2

2.3.1 True: (i), (ii), (v).

2.3.2 **(i)** The center of the coin must be farther from the borderline between tiles than $b/2$, so the probability of such event is $(b-a)^2/a^2$. **(ii)** The center of the coin must be closer to one of the corners than $b/2$, so the probability of such event is $\pi(b/2)^2/a^2$.

2.3.3 **(i)** The equation has two distinct roots if $b^2 - 4a > 0$, hence $a < b^2/4$. Consequently, the probability is $[2 + \int_{-1}^{1} b^2/(4db)]/4 = 13/24$. **(ii)** There are ten possible locations of the rectangle $A_1 \leq a \leq A_2, B_1 \leq b \leq B_2$ with respect to the parabola $a = b^2/4$. In each case, the probability of two roots is the ratio of the area of the rectangle below the parabola to the area $(A_2 - A_1)(B_2 - B_1)$ of the whole rectangle.

2.3.4 To compare the first and the third device, observe that y is also the distance from the center of the circle to the center of the chord. Thus the formula $y = \cos \alpha$ provides an explanation of lack of equivalence of devices that choose angle α and distance y to the center. To compare the second and the third mechanism of choosing a "random chord" note that the choice of the center of the chord *on a particular diameter* (scheme 2), and choice of the center as *a point in the interior of the circle* are not equivalent. To see that, notice that the probability of the distance from the center lying between y and $y+\delta y$ is δy for scheme 2, and $(\pi(y+\delta y)^2 - \pi y^2)/\pi \approx 2y\delta y$ for the scheme 3. Again, scheme 3 will tend to produce larger values of y, hence shorter chords.

2.4.1 We have $P(A) = 2P(B) = 3P(C) = 0.5(1 - P(A))$. It gives $P(A) = 1/3$, hence $P(B) = 1/6$ and $P(C) = 1/9$. Since $B \subset A$ and $A \cap C = \emptyset$, we obtain $B \cap C = \emptyset$, so that $P(B \cap C) = 0$. Consequently, $P(B \cup C) = P(B) + P(C) = 1/6 + 1/9 = 5/18$.

2.4.2 $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) = P(A) + P(B) + P(C) - P(B \cap C) = x + x + x - x/3 = 8x/3$.

2.4.3 $P(A \cup B^c) = P(A) + P(B) - P(A \cap B^c) = 1/2 + 1/2 - P(A \cap B^c) = 1 - [P(B^c) - P(A^c \cap B^c)] = 1 - [1/2 - 1/3] = 1/6$.

2.4.4 By the De Morgan's Law, $P[(A^c \cap B^c)^c] = P(A \cup B) = P(A) + P(B) - P(A \cap B) = 1 - P(A^c) + 1 - P(B^c) - P(A \cap B)$ so that $x = P(A \cap B)$.

2.4.5 In this and similar problems, it is perhaps best to use Venn diagrams, and fill the probabilities starting from the intersection of highest multiplicity. **(i)** $P(B) = 0.3 + 0.2 + 0.1 = 0.6$. **(ii)** $P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C) = 0.3 + 0 + 0.1 = 0.4$. **(iii)** $P(A \cap B^c \cap C^c) + P(A^c \cap B \cap C^c) + P(A^c \cap B^c \cap C) = 0.3 + 0.2 + 0 = 0.5$.

2.4.6 $P(A \cup B \cup C \cup D \cup E) = P(A) + P(B) + P(C) + P(D) + P(E) - P(A \cap B) - P(A \cap C) - \cdots - P(D \cap E) = 5k - 10p$ since all triple and higher order intersections are empty. **(i)** $1 = 5k - 10p = 5 \times 0.3 - 10p$, which gives $p = 0.05$. **(ii)** $1 \geq 5k - 10p = 5k - 0.1$, which gives $k \leq 0.22$. $5k - 10p \geq 0$ therefore $k \geq 0.02$.

2.4.7 **(i)** 0.1, **(ii)** 0.3, **(iii)** 1, **(iv)** 0.6.

2.4.8 With the obvious notation M, J, there are four possibilities: $M \cap J, M^c \cap J, M \cap J^c, M^c \cap J^c$. We have $P(M \cap J) = 0.32$, $P(J) = P(M \cap J) + P(M^c \cap J) = 0.4$, which gives $P(M^c \cap J) = 0.08$. Finally, $P(M^c) = 1 - P(M) = 1 - [P(M \cap J) + P(M \cap J^c)] = 0.2$ which implies $1 - 0.32 - P(M \cap J^c) = 0.2$, or $P(M \cap J^c) = 0.48$. The answers are : **(i)** $P(M \cap J^c) + P(M^c \cap J) = 0.48 + 0.08 = 0.56$; **(ii)** $P(M^c \cap J^c) = 1 - P(M \cap J) - P(M^c \cap J) - P(M \cap J^c) = 0.12$.

2.4.9 Let R and C denote the events "the phone returns the coin" and "the phone connects you with the number you dial". $P(R^c \cap C^c) = 0.3$, and consequently (using the fact that $P(R^c) = 1 - P(R) = 0.4$) we obtain $P(R^c) = P(R^c \cap C) + P(R^c \cap C^c) = 0.4$, so that $P(R^c \cap C) = 0.4 - 0.3 = 0.1$, $P(C) = P(R \cap$

$C) + P(R^c \cap C) = 0.2$, and $P(R \cap C) = P(\text{you talk for free}) = 0.2 - 0.1 = 0.1$.

2.4.10 Using notation of the solution to Problem 2.4.9, $P(R \cap C) = c$, so that $P(R) = a = P(R \cap C) + P(R \cap C^c)$, which gives $P(R \cap C^c) = a - c$. Similarly, $P(C) = b = P(R \cap C) + P(R^c \cap C)$, which gives $P(R^c \cap C) = b - c$. Consequently, $P(R^c \cap C^c) = 1 - c - (b - c) - (a - c) = 1 + c - a - b$.**(i)** The phone is individually honest if $P(R^c \cap C^c) = P(R \cap C) = 0$ which gives $c = 0, a + b = 1$. **(ii)** The phone is socially honest if $P(R^c) = P(C)$, which means $a + b = 1$.

2.4.11 With the obvious notation, $P(A^c \cap B^c) = 0.12$, hence $P(A \cap B^c) = P(B^c) - P(A^c \cap B^c) = 0.3 - 0.12 = 0.18$. Similarly, $P(A^c \cap B) = P(A^c) - P(A^c \cap B^c) = 0.5 - 0.12 = 0.38$. This gives $P(A \cap B) = 1 - (0.12 + 0.18 + 0.38) = 0.32$. The answers are: **(i)** $P(A \cap B^c) + P(A^c \cap B) = 0.18 + 0.38 = 0.56$. **(ii)** $1 - P(A^c \cap B^c) = 0.88$. **(iii)** $P(A \cap B) = 0.32$.

2.4.12 16/36=4/9. The outcomes whose sum is at least 10 are (ordinary die listed as first coordinate): (1, 9), (2, 8), (2, 9), (3, 8), (3, 9), (4, 6), (4, 8), (4, 9), (5, 5), (5, 6), (5, 8), (5, 9), (6, 5), (6, 6), (6, 8), (6, 9).

2.4.13 Let the probability of j dots be $jC, j = 1, \ldots, 6$. Then $C + 2C + \cdots + 6C = 1$ which gives $C = 1/21$. The answer is $1/21 + 3/21 + 5/21 = 3/7$.

2.5.1 We have, in binary expansion, $1,000,000 = 2^6 = 64$ and $1,011,111 = 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 95$. Consequently, to have $64 \leq X \leq 95$ the first two tosses must give HT, and the results of the subsequent five tosses may be arbitrary. This gives $2^5 = 32$ sequences with $4 \leq X \leq 95$, hence $P(64 \leq X \leq 95) = 32/128 = 1/4$.

2.5.2 **(i)** There are five choices with $X_1 = 6$, four choices with $X_1 = 5$ etc., so that $P\{X_1 > X_2 = X_3\} = (5 + 4 + 3 + 2 + 1)/216 = 15/216$. **(ii)** The number of sequences $X_1 < X_2 < X_3$ may be obtained as follows. If $X_1 = 1$ and $X_2 = 2$, then X_3 may be chosen in 4 ways. If $X_1 = 1, X_2 = 3$, then X_3 may be chosen in 3 ways, etc. This gives $4 + 3 + 2 + 1 = 10$ ways for choice of $X_2 < X_3$ if $X_1 = 1$. If $X_1 = 2$, similar argument gives $3 + 2 + 1 = 6$ choices for $X_2 < X_3$, and so on. This gives the total number of choices for $X_1 < X_2 < X_3$ as $(4 + 3 + 2 + 1) + (3 + 2 + 1) + (2 + 1) + 1 = 20$, hence $P\{X_1 < X_2 < X_3) = 20/216$. **(iii)** The direct enumeration of choices with $\max(X_1, X_2, X_3) = 3$ is cumbersome. The following tricks are useful here: $P\{\max(X_1, X_2, X_3) \leq k\} = k^3/216$, since $\max(X_1, X_2, X_3) \leq k$ if $X_1 \leq k, X_2 \leq k, X_3 \leq k$, which gives k^3 choices. We have now $P\{\max(X_1, X_2, X_3) = 3\} = P\{\max(X_1, X_2, X_3) \leq 3\} - P\{\max(X_1, X_2, X_3) \leq 2\} = (3^3 - 2^3)/216 = 19/216$. **(iv)** As in (iii), we have $P\{\min(X_1, X_2, X_3) \geq k\} = (7 - k)^3/216$, so that $P\{\min(X_1, X_2, X_3) = 2\} = P\{\min(X_1, X_2, X_3) \geq 2\} - P\{\min(X_1, X_2, X_3) \geq 3\} = (5^3 - 4^3)/216 = 61/216$.

2.5.3 $P(A_2 \cup A_3 \cup A_5 \cup A_7) = P(A_2) + P(A_3) + P(A_5) + P(A_7) - P(A_2 \cap A_3) -$

$P(A_2 \cap A_5) - P(A_2 \cap A_7) - P(A_3 \cap A_5) - P(A_3 \cap A_7) - P(A_5 \cap A_7) + P(A_2 \cap A_3 \cap A_5) + P(A_2 \cap A_3 \cap A_7) + P(A_2 \cap A_5 \cap A_7) + P(A_3 \cap A_5 \cap A_7) - P(A_2 \cap A_3 \cap A_5 \cap A_7)$. The consecutive terms are obtained from counting the numbers of n with $1 \leq n \leq 100$ which are divisible by the corresponding combination of primes 2, 3, 5 and 7. Thus gives $p = P(A_2 \cup A_3 \cup A_5 \cup A_7) = (50+33+20+14)/100 - (16+10+7+6+4+2)/100 + (3+2+1+0)/100 - 0/100 = 0.78$ Thus, there are $100 \times (1-p) = 22$ numbers which are not divisible by 2, 3, 5, or 7. One of these numbers is 1, which is not considered a prime. The remaining $100(1-p) - 1 = 21$ numbers are all primes, since for any composite number n between 2 and 100 the smallest prime factor must be 2, 3, 5 or 7 (any composite number whose smallest prime factor is 11 or more must be at least $11^2 = 121$). We have to include four primes 2, 3, 5 and 7, which gives $100(1-p) - 1 + 4 = 25$ primes not exceeding 100. Direct check gives: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

2.5.4 The series from which X is selected is of the form $4 + 5k, k = 0, \ldots, 99$, so that the last term is $4 + 5 \times 99 = 499$. The series from which Y is selected is of the form $1 + 4m, m = 0, 1, \ldots, 99$, so that the last term is 397. We need to find the number of terms common to both sequences. We must have $4 + 5k \leq 397$, which means that $k \leq 78.6$. If $4 + 5k = 1 + 4m$ then $4m = 5k + 3$, so that $5k + 3$ must be divisible by 4. Using the fact that $5 = 4 + 1$, we may write $4m = (4 + 1)k + 3 = 4k + k + 3$, so that $k + 3$ must be divisible by 4. This implies $k = 4r + 1, r = 0, 1, \ldots$. The largest value of r is obtained from $k = 4r + 1 \leq 78$, which gives $r \leq 19.25$, so that we have only 20 pairs with $X = Y$, obtained for $r = 0, 1, \ldots, 19$. There are $100 \times 100 = 10^4$ pairs (X, Y), each with the same probability, so $P(X = Y) = 20/10^4 = 0.002$.

2.7.1 Peter participates in the following lottery: with probability 1/8 he receives ticket T and pays \$25. With probability $(7/8) \times (1/4)$ he receives T and loses $\$25 + \$50 = \$75$. With probability $(7/8) \times (1/4) \times (1/2)$ he receives T and loses \$175. Finally, with probability $(7/8) \times (1/4) \times (1/2)$ he loses \$175. Since utility in money is assumed linear, we may put $u(\$x) = x$. Now, Tom agrees to the rules of the game, hence he must consider its expected utility (SEU) positive. Thus $(1/8)(u(T) - 25) + (7/32)(u(T) - 75) + (7/64)(-175) > 0$. This gives $u(T) > 200$. The ticket is worth more than \$200 to Peter.

2.7.2 Yes, Tom's utility of ticket satisfies $0.5(50 - u(T)) + 0.5 \times 50 > 0$, which gives $u(T) < 100$. Thus, \$150 is above the value of the ticket for Tom, and below the value for Peter.

2.7.3 Suppose that utilities (for Tom) of \$5,000,000, \$1,000,000 and \$0 are $A > B > C$. Without loss of generality, we may assume $B = 1, C = 0$ (so that A expresses utility of \$5,000,000 on a scale which has zero at \$0, and unit equal utility of \$1,000,000. Since $O1$ is better than $O2$, we must have $1 > 0.1A + 0.89$, hence $A < 0.11/0.1 = 1.1$. On the other hand, since $O3$ is better than $O4$, we must have $0.1A > 0.11$ which means that $A > 1.1$. Thus, the assumption that

Tom has utilities of the three outcomes compatible with his performances, leads to a contradiction.

CHAPTER 3

3.2.1 **(i)** Since $x(x-1) = 90$, $x_1 = 10$ or $x_2 = -9$. Only x_1 is the solution of the problem. **(ii)** Here $x(x-1)(x-2) = 10x(x-1)$. Since $x > 3$, we obtain $x - 2 = 10$, or $x = 12$.

3.2.2 The exact value (see Birthday Problem) is $p = P^5_{39}/39^5 = 0.76577$. Similarly $p = (38/39) \times (37/39) \times (36/39) \times (35/39) = (1-1/39) \times \cdots \times (1-4/39) \approx 1-(1+2+3+4)/39) = 1-(10/39) = 0.74359$. Using the formula $\log(1-x) \approx -x$ (x small) we obtain an analogy to (3.7), $p \approx e^{-(10/39)} = 0.77382$.

3.2.3 **(i)** $P^2_5 = 5 \times 4 = 20$. **(ii)** $P^2_4 + 1 = 4 \times 3 + 1 = 13$ (P^2_4 pairs can be formed out of four letters A, L, O, H, and one pair is AA). **(iii)** $P^2_4 + 2 = 4 \times 3 + 2 = 14$, two additional pairs being EE and TT.

3.2.4 In the product $1 \times 2 \times \cdots \times n = n!$ the number of 0's at the end equals the number of factors $10 = 2 \cdot 5$. Since the number of factors 2 exceeds the number of factors 5, the number of 0's is simply the same as the total number of factors 5 in $n!$. For 16! the number of factors 5 is 3, for 27! it is 6 (remember that $25 = 5^2$).

3.2.5 In all cases the total number of possible allocations of seats to n persons is $n!$. The process of determining the allocations of seats can be regarded as the result of three consecutive operations: 1. Selection of a pair of seats; 2. Allocating one of the selected seats to John and another to Mary; 3. Seating the remaining $n-2$ persons in the remaining seats. **(i)** There are $n-1$ ways of choosing a pair of neighboring seats, 2 ways of seating Mary and John on them, and $(n-2)!$ ways of seating the remaining persons, so that $p = (n-1)2(n-2)!/n! = 2/n$. **(ii)** As above, except the number of choices in (b) is 1, so $p = 1/n$. **(iii)** The number of choices in (a) is now $n(n-1)/2$, while the number of choices in (b) is 1. Thus, $p = [n(n-1)/2](n-2)!/n! = 1/2$. A simpler argument is that to each permutation of n persons (with John and Mary among them) one can assign another permutation by switching the places of John and Mary. This establishes a one–to– one mapping of the set of all permutations, and shows that in exactly half of them John sits to the right of Mary. **(iv)** Now the number of choices in (a) is $n-3$, so that (similarly as in (i)): $p = (n-3)2(n-2)!/n! = 2(n-3)/[n(n-1)]$.

3.2.6 Use the same three steps 1-3 as in Problem 3.2.5. **(i)** There are n pairs of neighboring seats in choice 1., so the answer to (i) is $p = n \cdot 2 \cdot (n-2)!/n! = 2/(n-1)$. (ii) $p = n \times 1 \times (n-2)!/n! = 1/(n-1)$. (iii) Answer requires specifying what does it mean "to sit on the right of someone" at a round table. (iv) There are n choices in step 1., hence $p = n \times 2 \times (n-2)!/n! = 2/(n-1)$. **(ii)** We have $n/2$ choices for step 1., so that $p = (n/2) \times [2(n-2)!/n!] = 1/n-1)$.

3.2.7 **(i)** $2 \times (5!)^2 = 28,800$ (5 men can be permuted in 5! ways, and the same holds for women; men can seat on seats $1, 3, 5, 7, 9$ or $2, 4, 6, 8, 10$). **(ii)** $6 \times (5!)^2 = 86,400$ (see solution to part (i); men can seat on chairs $1, 3, 5, 7, 9$; $1, 3, 5, 7, 10$; $1, 3, 5, 8, 10$; $\ldots$; $2, 4, 6, 8, 10$).

3.2.8 **(i)** $12 \times 17 = 204$. **(ii)** 3 h 46 min (18 periods of 12 minutes + 1 period of 10 minutes). **(iii)** 3 h 22 min. (16 periods of 12 minutes + 1 period of 10 minutes).

3.2.9 **(i)** There are $3(2+1)$ possibilities for feet and head. They can be combined with $5+3\times 4$ possibilities for the body. Altogether there are $3(2+1)(5+3\times 4) = 153$ combinations. **(ii)** If she buys a dress, the number of combinations increases to $3(2+1)(6+3\times 4) = 162$. If she buys a hat, the number of combinations increases to $3(3+1)(5+3\times 4) = 204$, so she should rather buy a hat. **(iii)** The number of non–matching combinations is: $1 \times (2+1) \times 2 + 3 \times 1 \times 1 \times 3 = 15$, so that the number of matching combinations is $153 - 15 = 138$.

3.2.10 **(i)** $5\times 3\times 15\times 3 = 675$. **(ii)** $675^2 = 455,625$. **(iii)** $5^2\times 3^2\times 15\times 14\times 3\times 2 = 283,500$.

3.2.11 $2\times 4\times\cdots\times(2n) = 2^n(1\times 2\times\cdots\times n) = 2^n n!$. Now $1\times 3\times\cdots\times(2n+1) = (2n+1)!/[2\times 4\times\cdots\times(2n)] = (2n+1)!/(2^n n!)$.

3.2.12 **(i)** Choose one person, say A, to have a multiple birthday; this can be done in r ways. Then allocate birthdays of the remaining $r-1$ persons. The probability of no repeated birthdays among them is p_{r-1}. Then allocate the birthday of A, joining him with one of the $r-1$, say with B. This can be done in $r-1$ ways, and leads to a configuration with exactly one pair of birthdays repeating. Each configuration is counted twice, since B could be chosen first and coupled with A. This gives $e_r = rp_{r-1}[(r-1)/2]$, as asserted. **(ii)** The total number of ways of allocating birthdays to r people is 365^r. To count the number of allocations with exactly one pair with a common birthday, note that: the pair with common birthday can be chosen in $r(r-1)/2$ ways; the birthday of the chosen pair can be selected in 365 ways; the birthdays of the remaining $r-2$ people can be selected in $P_{364}^{r-2} = 364(364-1)\times\cdots\times(364-(r-2)+1) = (365-1)(365-2)\times\cdots\times(365-r+2)$ ways. The formula expressing e_r through p_r can be now obtained by simple algebra.

3.2.13 The first digit can be chosen in 9 ways; second also in 9 ways, third in 8 ways; $9 \times 9 \times 8 = 648$. To count the odd integers, observe that the last digit can be chosen in 5 ways. If the middle digit is 0, the first can be chosen in 8 ways, otherwise in 7 ways. Answer: $5 \times 8 \times 7 + 5 \times 1 \times 8 = 320$.

3.2.14 If $n = 2m$ then the majority is at least $m+1$ votes, and A is a pivot if it is on $(m-1)$st or mth place in the permutation. If $n = 2m+1$, then the