

2.1 Let

$$S_x B \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and write the S_x eigenvalue equations in matrix notation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = +\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which yields

$$\begin{aligned} a + \beta &= +\frac{\hbar}{2} & \chi + \delta &= +\frac{\hbar}{2} \\ \alpha - \beta &= -\frac{\hbar}{2} & \chi - \delta &= +\frac{\hbar}{2} \end{aligned}$$

Solve by adding and subtracting the equations to get

$$a = 0 \quad \beta = \frac{\hbar}{2} \quad \chi = \frac{\hbar}{2} \quad \delta = 0$$

Hence the matrix representing S_x in the S_z basis is

$$S_x B \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Let

$$S_y B \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and write the S_y eigenvalue equations in matrix notation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = +\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = -\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

which yields

$$\begin{aligned} a + i\beta &= +\frac{\hbar}{2} & \chi + i\delta &= +i\frac{\hbar}{2} \\ \alpha - i\beta &= -\frac{\hbar}{2} & \chi - i\delta &= +i\frac{\hbar}{2} \end{aligned}$$

Solve by adding and subtracting the equations to get

$$a = 0 \quad \beta = -i\frac{\hbar}{2} \quad \chi = i\frac{\hbar}{2} \quad \delta = 0$$

Hence the matrix representing S_y in the S_z basis is

$$S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

2.2 Solve the secular equation

$$\det[S_x - \lambda I] = 0$$

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0$$

Solve to find the eigenvalues

$$\lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$$

$$\lambda^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\lambda = \pm \frac{\hbar}{2}$$

which was to be expected, because we know that the only possible results of a measurement of any spin component are $\pm \hbar/2$. Find the eigenvectors. For the positive eigenvalue:

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = +\frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

yields

$$\beta = \alpha$$

The normalization condition yields

$$|\alpha|^2 + |\alpha|^2 = 1$$

$$|\alpha|^2 = \frac{1}{2}$$

Choose α to be real and positive, resulting in

$$\alpha = \frac{1}{\sqrt{2}}$$

$$\beta = \frac{1}{\sqrt{2}}$$

so the eigenvector corresponding to the positive eigenvalue is

$$|+\rangle_x \text{ B } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Likewise, the eigenvector for the negative eigenvalue is

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$b = -\alpha$$

The normalization condition yields

$$|a|^2 + |b|^2 = 1$$

$$|a|^2 = \frac{1}{2}$$

Choose a to be real and positive, resulting in

$$a = \frac{1}{\sqrt{2}}$$

$$\beta = -\frac{1}{\sqrt{2}}$$

so the eigenvector corresponding to the negative eigenvalue is

$$|-\rangle_x \text{ B } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2.3 From Eq. (1.37), we know the S_z eigenstates in the S_x basis:

$$|+\rangle = \frac{1}{\sqrt{2}}(|+\rangle_x + |-\rangle_x)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|+\rangle_x - |-\rangle_x)$$

Let the representation of S_z in the S_x basis be

$$S_z \text{ B } \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and write the S_z eigenvalue equations in matrix notation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = +\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

These yield

$$\begin{aligned} a + \beta &= +\frac{\eta}{2} & \chi + \delta &= +\frac{\eta}{2} \\ \alpha - \beta &= -\frac{\eta}{2} & \chi - \delta &= +\frac{\eta}{2} \end{aligned}$$

Solve by adding and subtracting the equations to get

$$a = 0 \quad \beta = \frac{\eta}{2} \quad \chi = \frac{\eta}{2} \quad \delta = 0$$

Hence the matrix representing S_z in the S_x basis is

$$S_z \mathbf{B}_{S_x} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now diagonalize:

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \Rightarrow \lambda = \pm \frac{\hbar}{2}$$

as expected. Find the eigenvectors:

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = +\frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \beta = \alpha$$

yielding

$$|+\rangle \mathbf{B}_{S_x} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Likewise

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \beta = -\alpha$$

$$|-\rangle \mathbf{B}_{S_x} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence the eigenvalue equations are

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = +\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \Sigma_z |+\rangle = +\frac{\hbar}{2} |+\rangle \quad OK$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \Sigma_z |-\rangle = -\frac{\hbar}{2} |-\rangle \quad OK$$

2.4 The general matrix is

$$A B \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The matrix elements are

$$\langle +|A|+ \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = a$$

$$\langle +|A|- \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = b$$

$$\langle -|A|+ \rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = c$$

$$\langle -|A|- \rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = d$$

Hence we get

$$A B \begin{pmatrix} \langle +|A|+ \rangle & \langle +|A|- \rangle \\ \langle -|A|+ \rangle & \langle -|A|- \rangle \end{pmatrix}$$

2.5 The commutators are

$$\begin{aligned} [S_x, S_y] B \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\eta}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{\eta}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\eta}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ B \left(\frac{\eta}{2} \right)^2 \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right] \\ B \left(\frac{\eta}{2} \right)^2 \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} B \eta \left(\frac{\eta}{2} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ = \eta \Sigma_z^{\eta} \end{aligned}$$

$$\begin{aligned} [S_y, S_z] B \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\eta}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\eta}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\eta}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ B \left(\frac{\eta}{2} \right)^2 \left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \\ B \left(\frac{\eta}{2} \right)^2 \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} B \eta \left(\frac{\eta}{2} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ = \eta \Sigma_x^{\eta} \end{aligned}$$

$$\begin{aligned}
[S_z, S_x] &= \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \\
&= \frac{\hbar^2}{4} \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \frac{\hbar^2}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \hbar S_y
\end{aligned}$$

2.6 The spin component operator S_n is

$$\begin{aligned}
S_n &= \mathbf{S} \cdot \hat{\mathbf{n}} \\
&= S_x \sin\theta \cos\phi + S_y \sin\theta \sin\phi + S_z \cos\theta
\end{aligned}$$

Using the matrix representations for S_x , S_y , and S_z gives

$$\begin{aligned}
S_n &= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin\theta \cos\phi + \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sin\theta \sin\phi + \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos\theta \\
&= \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta \cos\phi - i \sin\theta \sin\phi \\ \sin\theta \cos\phi + i \sin\theta \sin\phi & -\cos\theta \end{pmatrix} \\
&= \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}
\end{aligned}$$

2.7 Diagonalize S_n :

$$S_n = \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

Now diagonalize:

$$\begin{vmatrix} \frac{\hbar}{2} \cos\theta - \lambda & \frac{\hbar}{2} \sin\theta e^{-i\phi} \\ \frac{\hbar}{2} \sin\theta e^{i\phi} & -\frac{\hbar}{2} \cos\theta - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - \left(\frac{\hbar}{2}\right)^2 \cos^2\theta - \left(\frac{\hbar}{2}\right)^2 \sin^2\theta = 0 \Rightarrow \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \Rightarrow \lambda = \pm \frac{\hbar}{2}$$

as expected. Find the eigenvectors:

$$\frac{\hbar}{2} \begin{pmatrix} \cos\theta & \alpha v \theta \varepsilon^{-\varphi} \\ \alpha v \theta \varepsilon^{\varphi} & -\chi \sigma \theta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = +\frac{\eta}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\alpha \chi \sigma \theta + \beta \alpha v \theta \varepsilon^{-\varphi} = \alpha \Rightarrow \beta = \alpha \varepsilon^{\varphi} \frac{1 - \chi \sigma \theta}{\alpha v \theta}$$

The normalization condition yields

$$|\alpha|^2 + |\alpha|^2 \left(\frac{1 - \chi \sigma \theta}{\alpha v \theta} \right)^2 = 1$$

$$|\alpha|^2 = \frac{\alpha v^2 \theta}{2 - 2 \chi \sigma \theta} = \frac{4 \alpha v^2 \frac{\theta}{2} \chi \sigma^2 \frac{\theta}{2}}{4 \alpha v^2 \frac{\theta}{2}} = \chi \sigma^2 \frac{\theta}{2}$$

yielding

$$|+\rangle_n = \cos \frac{\theta}{2} |+\rangle + e^{i\varphi} \sin \frac{\theta}{2} |-\rangle$$

Likewise

$$\frac{\hbar}{2} \begin{pmatrix} \cos\theta & \alpha v \theta \varepsilon^{-\varphi} \\ \alpha v \theta \varepsilon^{\varphi} & -\chi \sigma \theta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\frac{\eta}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\alpha \chi \sigma \theta + \beta \alpha v \theta \varepsilon^{-\varphi} = -\alpha \Rightarrow \beta = -\alpha \varepsilon^{\varphi} \frac{1 + \chi \sigma \theta}{\alpha v \theta}$$

The normalization condition yields

$$|\alpha|^2 + |\alpha|^2 \left(\frac{1 + \chi \sigma \theta}{\alpha v \theta} \right)^2 = 1$$

$$|\alpha|^2 = \frac{\alpha v^2 \theta}{2 + 2 \chi \sigma \theta} = \frac{4 \alpha v^2 \frac{\theta}{2} \chi \sigma^2 \frac{\theta}{2}}{4 \chi \sigma^2 \frac{\theta}{2}} = \alpha v^2 \frac{\theta}{2}$$

yielding

$$|-\rangle_n = \sin \frac{\theta}{2} |+\rangle - e^{i\varphi} \cos \frac{\theta}{2} |-\rangle$$

2.8 The $|+\rangle_n$ eigenstate is

$$|+\rangle_n = \cos \frac{\theta}{2} |+\rangle + e^{i\varphi} \sin \frac{\theta}{2} |-\rangle = \cos \frac{\pi}{8} |+\rangle + e^{i5\pi/3} \sin \frac{\pi}{8} |-\rangle$$

The probabilities are

$$\begin{aligned}
\mathcal{P}_{+y} &= \left| \langle + | + \rangle_n \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{i}{\sqrt{2}} \langle - | \right) (\cos \frac{\pi}{8} | + \rangle + e^{i5\pi/3} \sin \frac{\pi}{8} | - \rangle) \right|^2 \\
&= \frac{1}{2} \left| \cos \frac{\pi}{8} - i e^{i5\pi/3} \sin \frac{\pi}{8} \right|^2 = \frac{1}{2} \left| \cos \frac{\pi}{8} + \sin \frac{\pi}{8} \sin \frac{5\pi}{3} - i \sin \frac{\pi}{8} \cos \frac{5\pi}{3} \right|^2 \\
&= \frac{1}{2} (\cos^2 \frac{\pi}{8} + \sin^2 \frac{\pi}{8} \sin^2 \frac{5\pi}{3} + \sin^2 \frac{\pi}{8} \cos^2 \frac{5\pi}{3} + 2 \cos \frac{\pi}{8} \sin \frac{\pi}{8} \sin \frac{5\pi}{3}) \\
&= \frac{1}{2} (1 + 2 \cos \frac{\pi}{8} \sin \frac{\pi}{8} \sin \frac{5\pi}{3}) \cong 0.194
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{-y} &= \left| \langle - | + \rangle_n \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{i}{\sqrt{2}} \langle - | \right) (\cos \frac{\pi}{8} | + \rangle + e^{i5\pi/3} \sin \frac{\pi}{8} | - \rangle) \right|^2 \\
&= \frac{1}{2} \left| \cos \frac{\pi}{8} + i e^{i5\pi/3} \sin \frac{\pi}{8} \right|^2 = \frac{1}{2} \left| \cos \frac{\pi}{8} - \sin \frac{\pi}{8} \sin \frac{5\pi}{3} + i \sin \frac{\pi}{8} \cos \frac{5\pi}{3} \right|^2 \\
&= \frac{1}{2} (\cos^2 \frac{\pi}{8} + \sin^2 \frac{\pi}{8} \sin^2 \frac{5\pi}{3} + \sin^2 \frac{\pi}{8} \cos^2 \frac{5\pi}{3} - 2 \cos \frac{\pi}{8} \sin \frac{\pi}{8} \sin \frac{5\pi}{3}) \\
&= \frac{1}{2} (1 - 2 \cos \frac{\pi}{8} \sin \frac{\pi}{8} \sin \frac{5\pi}{3}) \cong 0.806
\end{aligned}$$

2.9 The expectation value of S_z is easy to do in Dirac notation:

$$\langle S_z \rangle = \langle + | S_z | + \rangle = \langle + | \frac{\hbar}{2} | + \rangle = \frac{\hbar}{2} \langle + | + \rangle = \frac{\hbar}{2}$$

The expectation values of S_x and S_y are easier in matrix notation:

$$\langle S_x \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\hbar$$

$$\langle S_y \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ i \end{pmatrix} = 0\hbar$$

To find the uncertainties, we need the expectation values of the squares:

$$\langle S_z^2 \rangle = \langle + | S_z^2 | + \rangle = \langle + | \frac{\hbar}{2} \frac{\hbar}{2} | + \rangle = \left(\frac{\hbar}{2} \right)^2 \langle + | + \rangle = \left(\frac{\hbar}{2} \right)^2$$

$$\langle S_x^2 \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{\hbar}{2} \right)^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{\hbar}{2} \right)^2$$

$$\langle S_y^2 \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{\hbar}{2} \right)^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{\hbar}{2} \right)^2$$

The uncertainties are

$$\Delta S_z = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - \left(\frac{\hbar}{2}\right)^2} = 0\hbar$$

$$\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \left(\frac{\hbar}{2}\right)$$

$$\Delta S_y = \sqrt{\langle S_y^2 \rangle - \langle S_y \rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \left(\frac{\hbar}{2}\right)$$

2.10 These expectation values are easier in matrix notation:

$$\langle S_x \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{4} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = 0\hbar$$

$$\langle S_y \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{4} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{2}$$

$$\langle S_z \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{4} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = 0\hbar$$

To find the uncertainties, we need the expectation values of the squares:

$$\langle S_x^2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2$$

$$\langle S_y^2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2$$

$$\langle S_z^2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2$$

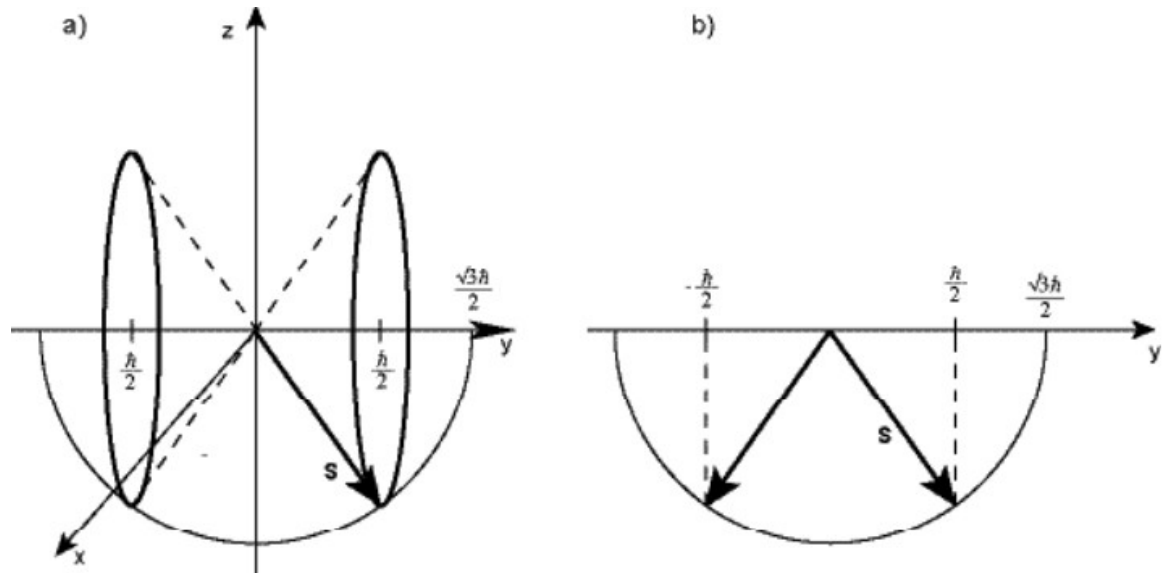
The uncertainties are

$$\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \left(\frac{\hbar}{2}\right)$$

$$\Delta S_y = \sqrt{\langle S_y^2 \rangle - \langle S_y \rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - \left(\frac{\hbar}{2}\right)^2} = 0\hbar$$

$$\Delta S_z = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \left(\frac{\hbar}{2}\right)$$

In the vector model, shown below, the spin is precessing around the y-axis at a constant angle such the y-component of the spin is constant and x- and z-components oscillate about zero.



2.11 The commutators in matrix notation are

$$\begin{aligned}
 [\mathbf{S}^2, S_x] &= B \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\eta}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\eta}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &= B \frac{3\eta^3}{8} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \\
 &= 0 \\
 [\mathbf{S}^2, S_y] &= B \frac{3\eta^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\eta}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{\eta}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{3\eta^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &= B \frac{3\eta^3}{8} \left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] \\
 &= 0 \\
 [\mathbf{S}^2, S_z] &= B \frac{3\eta^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\eta}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\eta}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{3\eta^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &= B \frac{3\eta^3}{8} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\
 &= 0
 \end{aligned}$$

In abstract notation, the commutators are

$$\begin{aligned}
[\mathbf{S}^2, S_x] &= [\Sigma_\xi^2 + \Sigma_\psi^2 + \Sigma_\zeta^2, \Sigma_\xi] = [\Sigma_\xi^2, \Sigma_\xi] + [\Sigma_\psi^2, \Sigma_\xi] + [\Sigma_\zeta^2, \Sigma_\xi] \\
&= \Sigma_\xi^2 \Sigma_\xi - \Sigma_\xi \Sigma_\xi^2 + \Sigma_\psi^2 \Sigma_\xi - \Sigma_\xi \Sigma_\psi^2 + \Sigma_\zeta^2 \Sigma_\xi - \Sigma_\xi \Sigma_\zeta^2 \\
&= \Sigma_\xi^3 - \Sigma_\xi^3 + \Sigma_\psi \Sigma_\psi \Sigma_\xi - \Sigma_\xi \Sigma_\psi \Sigma_\psi + \Sigma_\zeta \Sigma_\zeta \Sigma_\xi - \Sigma_\xi \Sigma_\zeta \Sigma_\zeta \\
&= 0 + \Sigma_\psi (\Sigma_\xi \Sigma_\psi - \eta \Sigma_\zeta) - (\Sigma_\psi \Sigma_\xi + \eta \Sigma_\zeta) \Sigma_\psi + \Sigma_\zeta (\Sigma_\xi \Sigma_\zeta + \eta \Sigma_\psi) - (\Sigma_\zeta \Sigma_\xi - \eta \Sigma_\psi) \Sigma_\zeta \\
&= \Sigma_\psi \Sigma_\xi \Sigma_\psi - \eta \Sigma_\psi \Sigma_\zeta - \Sigma_\psi \Sigma_\xi \Sigma_\psi - \eta \Sigma_\zeta \Sigma_\psi + \Sigma_\zeta \Sigma_\xi \Sigma_\zeta + \eta \Sigma_\zeta \Sigma_\psi - \Sigma_\zeta \Sigma_\xi \Sigma_\zeta + \eta \Sigma_\psi \Sigma_\zeta \\
&= 0 \\
[\mathbf{S}^2, \Sigma_\psi] &= [\Sigma_\xi^2 + \Sigma_\psi^2 + \Sigma_\zeta^2, \Sigma_\psi] = [\Sigma_\xi^2, \Sigma_\psi] + [\Sigma_\psi^2, \Sigma_\psi] + [\Sigma_\zeta^2, \Sigma_\psi] \\
&= \Sigma_\xi^2 \Sigma_\psi - \Sigma_\psi \Sigma_\xi^2 + \Sigma_\psi^2 \Sigma_\psi - \Sigma_\psi \Sigma_\psi^2 + \Sigma_\zeta^2 \Sigma_\psi - \Sigma_\psi \Sigma_\zeta^2 \\
&= \Sigma_\xi \Sigma_\xi \Sigma_\psi - \Sigma_\psi \Sigma_\xi \Sigma_\xi + \Sigma_\psi^3 - \Sigma_\psi^3 + \Sigma_\zeta \Sigma_\zeta \Sigma_\psi - \Sigma_\psi \Sigma_\zeta \Sigma_\zeta \\
&= \Sigma_\xi (\Sigma_\psi \Sigma_\xi + \eta \Sigma_\zeta) - (\Sigma_\xi \Sigma_\psi - \eta \Sigma_\zeta) \Sigma_\xi + 0 + \Sigma_\zeta (\Sigma_\psi \Sigma_\zeta - \eta \Sigma_\xi) - (\Sigma_\zeta \Sigma_\psi + \eta \Sigma_\xi) \Sigma_\zeta \\
&= \Sigma_\xi \Sigma_\psi \Sigma_\xi + \eta \Sigma_\xi \Sigma_\zeta - \Sigma_\xi \Sigma_\psi \Sigma_\xi + \eta \Sigma_\zeta \Sigma_\xi + \Sigma_\zeta \Sigma_\psi \Sigma_\zeta - \eta \Sigma_\zeta \Sigma_\xi - \Sigma_\zeta \Sigma_\psi \Sigma_\zeta - \eta \Sigma_\xi \Sigma_\zeta \\
&= 0 \\
[\mathbf{S}^2, \Sigma_\zeta] &= [\Sigma_\xi^2 + \Sigma_\psi^2 + \Sigma_\zeta^2, \Sigma_\zeta] = [\Sigma_\xi^2, \Sigma_\zeta] + [\Sigma_\psi^2, \Sigma_\zeta] + [\Sigma_\zeta^2, \Sigma_\zeta] \\
&= \Sigma_\xi^2 \Sigma_\zeta - \Sigma_\zeta \Sigma_\xi^2 + \Sigma_\psi^2 \Sigma_\zeta - \Sigma_\zeta \Sigma_\psi^2 + \Sigma_\zeta^2 \Sigma_\zeta - \Sigma_\zeta \Sigma_\zeta^2 \\
&= \Sigma_\xi \Sigma_\xi \Sigma_\zeta - \Sigma_\zeta \Sigma_\xi \Sigma_\xi + \Sigma_\psi \Sigma_\psi \Sigma_\zeta - \Sigma_\zeta \Sigma_\psi \Sigma_\psi + \Sigma_\zeta^3 - \Sigma_\zeta^3 \\
&= \Sigma_\xi (\Sigma_\zeta \Sigma_\xi - \eta \Sigma_\psi) - (\Sigma_\zeta \Sigma_\xi + \eta \Sigma_\psi) \Sigma_\xi + \Sigma_\psi (\Sigma_\zeta \Sigma_\psi + \eta \Sigma_\xi) - (\Sigma_\psi \Sigma_\zeta - \eta \Sigma_\xi) \Sigma_\psi + 0 \\
&= \Sigma_\xi \Sigma_\zeta \Sigma_\xi - \eta \Sigma_\xi \Sigma_\psi - \Sigma_\zeta \Sigma_\xi \Sigma_\xi - \eta \Sigma_\psi \Sigma_\xi + \Sigma_\psi \Sigma_\zeta \Sigma_\psi + \eta \Sigma_\psi \Sigma_\xi - \Sigma_\psi \Sigma_\zeta \Sigma_\psi + \eta \Sigma_\xi \Sigma_\psi \\
&= 0
\end{aligned}$$

2.12 For S_x the diagonalization yields the eigenvalues

$$\begin{aligned}
S_x &= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\begin{pmatrix} -\lambda & \frac{\eta}{\sqrt{2}} & 0 \\ \frac{\eta}{\sqrt{2}} & -\lambda & \frac{\eta}{\sqrt{2}} \\ 0 & \frac{\eta}{\sqrt{2}} & -\lambda \end{pmatrix} &= 0 \Rightarrow -\lambda(\lambda^2 - \frac{\eta^2}{2}) - \frac{\eta}{\sqrt{2}}(-\lambda \frac{\eta}{\sqrt{2}}) = 0 \\
\lambda(\lambda^2 - \eta^2) &= 0 \Rightarrow \lambda = \eta, 0, -\eta
\end{aligned}$$

and the eigenvectors

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} = 1\eta \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} \Rightarrow \begin{aligned} \beta &= \alpha\sqrt{2} \\ \alpha + \chi &= \beta\sqrt{2} \\ \beta &= \chi\sqrt{2} \end{aligned}$$

$$|\alpha|^2 + |\beta|^2 + |\chi|^2 = 1 \Rightarrow |\beta|^2 (\frac{1}{2} + 1 + \frac{1}{2}) = 1 \Rightarrow \beta = \frac{1}{\sqrt{2}}, \alpha = \frac{1}{2}, \chi = \frac{1}{2}$$

$$|1\rangle_{\xi} = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} = 0\eta \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} \Rightarrow \begin{aligned} \beta &= 0 \\ \alpha + \chi &= 0 \\ \beta &= 0 \end{aligned}$$

$$|\alpha|^2 + |\beta|^2 + |\chi|^2 = 1 \Rightarrow |\alpha|^2 (1+1) = 1 \Rightarrow \alpha = \frac{1}{\sqrt{2}}, \beta = 0, \chi = -\frac{1}{\sqrt{2}}$$

$$|0\rangle_{\xi} = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|-1\rangle$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} = -1\eta \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} \Rightarrow \begin{aligned} \beta &= -\alpha\sqrt{2} \\ \alpha + \chi &= -\beta\sqrt{2} \\ \beta &= -\chi\sqrt{2} \end{aligned}$$

$$|\alpha|^2 + |\beta|^2 + |\chi|^2 = 1 \Rightarrow |\beta|^2 (\frac{1}{2} + 1 + \frac{1}{2}) = 1 \Rightarrow \beta = -\frac{1}{\sqrt{2}}, \alpha = \frac{1}{2}, \chi = \frac{1}{2}$$

$$|-1\rangle_{\xi} = \frac{1}{2}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle$$

For S_y the diagonalization yields

$$S_y B \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\begin{pmatrix} -\lambda & \frac{-m}{\sqrt{2}} & 0 \\ \frac{m}{\sqrt{2}} & -\lambda & \frac{-m}{\sqrt{2}} \\ 0 & \frac{m}{\sqrt{2}} & -\lambda \end{pmatrix} = 0 \Rightarrow -\lambda(\lambda^2 - \frac{m^2}{2}) - \frac{-m}{\sqrt{2}}(-\lambda \frac{m}{\sqrt{2}}) = 0$$

$$\lambda(\lambda^2 - m^2) = 0 \Rightarrow \lambda = 0, m, -m$$

and the eigenvectors

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} = 1\eta \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} \Rightarrow \begin{aligned} -i\beta &= \alpha\sqrt{2} \\ i\alpha - i\chi &= \beta\sqrt{2} \\ i\beta &= \chi\sqrt{2} \end{aligned}$$

$$|\alpha|^2 + |\beta|^2 + |\chi|^2 = 1 \Rightarrow |\beta|^2 (\frac{1}{2} + 1 + \frac{1}{2}) = 1 \Rightarrow \beta = \frac{1}{\sqrt{2}}, \alpha = \frac{1}{2}, \chi = -\frac{1}{2}$$

$$|1\rangle_{\psi} = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} = 0 \eta \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} \Rightarrow \begin{aligned} -i\beta &= 0 \\ i\alpha - i\chi &= 0 \\ i\beta &= 0 \end{aligned}$$

$$|\alpha|^2 + |\beta|^2 + |\chi|^2 = 1 \Rightarrow |\alpha|^2(1+1) = 1 \Rightarrow \alpha = \frac{1}{\sqrt{2}}, \beta = 0, \chi = \frac{1}{\sqrt{2}}$$

$$|0\rangle_\psi = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|-1\rangle$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} = -1 \eta \begin{pmatrix} \alpha \\ \beta \\ \chi \end{pmatrix} \Rightarrow \begin{aligned} -i\beta &= -\alpha\sqrt{2} \\ i\alpha - i\chi &= -\beta\sqrt{2} \\ i\beta &= -\chi\sqrt{2} \end{aligned}$$

$$|\alpha|^2 + |\beta|^2 + |\chi|^2 = 1 \Rightarrow |\beta|^2(\frac{1}{2} + 1 + \frac{1}{2}) = 1 \Rightarrow \beta = -\frac{i}{\sqrt{2}}, \alpha = \frac{1}{2}, \chi = -\frac{1}{2}$$

$$|-1\rangle_\psi = \frac{1}{2}|1\rangle - \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle$$

2.13 The commutators are

$$[S_x, S_y] B \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} - \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B \frac{\eta^2}{2} \left[\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix} \right]$$

$$B \frac{\eta^2}{2} \begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix} B m \eta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= m \eta \Sigma_z$$

$$[S_y, S_z] B \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \eta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \eta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$B \frac{\eta^2}{\sqrt{2}} \left[\begin{pmatrix} 0 & 0 & 0 \\ i & 0 & i \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix} \right]$$

$$B \frac{\eta^2}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix} B m \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= m \eta \Sigma_x$$

$$\begin{aligned}
& [S_z, S_x] B \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \eta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
& B \frac{\eta^2}{\sqrt{2}} \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \right] \\
& B \frac{\eta^2}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} B \eta \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
& = \eta \Sigma_\psi
\end{aligned}$$

2.14 Using the component matrices we find

$$\begin{aligned}
\mathbf{S}^2 &= \Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2 \\
& B \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
& + \eta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \eta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
& B \frac{\eta^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\eta^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \eta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
& B 2\eta^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

The eigenvalue equation is



For spin-1 this is

$$\mathbf{S}^2 |1m\rangle = 2\eta^2 |1\mu\rangle$$

Hence the $\mathbf{S}^2$ operator must be $2\hbar^2$ times the identity matrix:

$$\mathbf{S}^2 B 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.15 a) The possible results of a measurement of the spin component S_z are always $+1\hbar$, $0\hbar$, $-1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}\mathcal{P}_1 &= \left| \langle 1 | \psi_{iv} \rangle \right|^2 = \left| \langle 1 | \left[\frac{2}{\sqrt{29}} |1\rangle + \frac{3i}{\sqrt{29}} |0\rangle - \frac{4}{\sqrt{29}} |-1\rangle \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \langle 1|1\rangle + \frac{3i}{\sqrt{29}} \langle 1|0\rangle - \frac{4}{\sqrt{29}} \langle 1|-1\rangle \right|^2 = \left| \frac{2}{\sqrt{29}} \right|^2 = \frac{4}{29} \\ \mathcal{P}_0 &= \left| \langle 0 | \psi_{iv} \rangle \right|^2 = \left| \langle 0 | \left[\frac{2}{\sqrt{29}} |1\rangle + \frac{3i}{\sqrt{29}} |0\rangle - \frac{4}{\sqrt{29}} |-1\rangle \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \langle 0|1\rangle + \frac{3i}{\sqrt{29}} \langle 0|0\rangle - \frac{4}{\sqrt{29}} \langle 0|-1\rangle \right|^2 = \left| \frac{3i}{\sqrt{29}} \right|^2 = \frac{9}{29} \\ \mathcal{P}_{-1} &= \left| \langle -1 | \psi_{iv} \rangle \right|^2 = \left| \langle -1 | \left[\frac{2}{\sqrt{29}} |1\rangle + \frac{3i}{\sqrt{29}} |0\rangle - \frac{4}{\sqrt{29}} |-1\rangle \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \langle -1|1\rangle + \frac{3i}{\sqrt{29}} \langle -1|0\rangle - \frac{4}{\sqrt{29}} \langle -1|-1\rangle \right|^2 = \left| -\frac{4}{\sqrt{29}} \right|^2 = \frac{16}{29}\end{aligned}$$

The three probabilities add to unity, as they must.

b) The possible results of a measurement of the spin component S_x are always $+1\hbar$, $0\hbar$, $-1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}\mathcal{P}_{1x} &= \left| \langle 1 | \psi_{iv} \rangle \right|^2 = \left| \left(\frac{1}{2} \langle 1 | + \frac{1}{\sqrt{2}} \langle 0 | + \frac{1}{2} \langle -1 | \right) \left(\frac{2}{\sqrt{29}} |1\rangle + \frac{3i}{\sqrt{29}} |0\rangle - \frac{4}{\sqrt{29}} |-1\rangle \right) \right|^2 \\ &= \left| \frac{1}{\sqrt{29}} + \frac{3i}{\sqrt{2}\sqrt{29}} - \frac{2}{\sqrt{29}} \right|^2 = \frac{1}{58} \left| -\sqrt{2} + 3i \right|^2 = \frac{11}{58} \\ \mathcal{P}_{0x} &= \left| \langle 0 | \psi_{iv} \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle 1 | - \frac{1}{\sqrt{2}} \langle -1 | \right) \left(\frac{2}{\sqrt{29}} |1\rangle + \frac{3i}{\sqrt{29}} |0\rangle - \frac{4}{\sqrt{29}} |-1\rangle \right) \right|^2 \\ &= \left| \frac{2}{\sqrt{2}\sqrt{29}} + \frac{4}{\sqrt{2}\sqrt{29}} \right|^2 = \frac{36}{58} \\ \mathcal{P}_{-1x} &= \left| \langle -1 | \psi_{iv} \rangle \right|^2 = \left| \left(\frac{1}{2} \langle 1 | - \frac{1}{\sqrt{2}} \langle 0 | + \frac{1}{2} \langle -1 | \right) \left(\frac{2}{\sqrt{29}} |1\rangle + \frac{3i}{\sqrt{29}} |0\rangle - \frac{4}{\sqrt{29}} |-1\rangle \right) \right|^2 \\ &= \left| \frac{1}{\sqrt{29}} - \frac{3i}{\sqrt{2}\sqrt{29}} - \frac{2}{\sqrt{29}} \right|^2 = \frac{1}{58} \left| -\sqrt{2} - 3i \right|^2 = \frac{11}{58}\end{aligned}$$

The three probabilities add to unity, as they must.

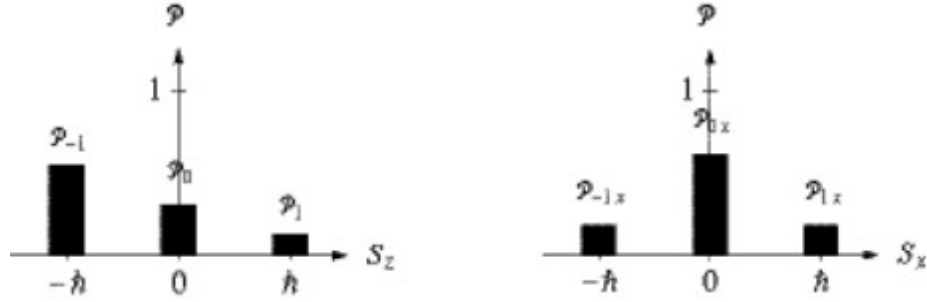
c) For the first measurement, the expectation value is

$$\langle S_z \rangle = \sum_{\mu} \mu \eta \Pi_{\mu} = 1\eta \frac{4}{29} + 0\eta \frac{9}{29} + (-1)\eta \frac{16}{29} = -\eta \frac{12}{29}$$

For the second measurement, the expectation value is

$$\langle S_x \rangle = \sum_{\mu} \mu \eta \Pi_{\mu\xi} = 1\eta \frac{11}{58} + 0\eta \frac{36}{58} + (-1)\eta \frac{11}{58} = 0\eta$$

The histograms are shown below.



2.16 a) The possible results of a measurement of the spin component S_z are always $+1\hbar$, $0\hbar$, $-1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}\mathcal{P}_1 &= |\langle 1 | \psi_{iv} \rangle|^2 = \left| \langle 1 | \left[\frac{2}{\sqrt{29}} |1\rangle_\psi + \frac{3i}{\sqrt{29}} |0\rangle_\psi - \frac{4}{\sqrt{29}} |-1\rangle_\psi \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \langle 1 | 1 \rangle_\psi + \frac{3i}{\sqrt{29}} \langle 1 | 0 \rangle_\psi - \frac{4}{\sqrt{29}} \langle 1 | -1 \rangle_\psi \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \frac{1}{2} + \frac{3i}{\sqrt{29}} \frac{1}{\sqrt{2}} - \frac{4}{\sqrt{29}} \frac{1}{2} \right|^2 = \frac{1}{58} |-\sqrt{2} + 3i|^2 = \frac{11}{58}\end{aligned}$$

$$\begin{aligned}\mathcal{P}_0 &= |\langle 0 | \psi_{iv} \rangle|^2 = \left| \langle 0 | \left[\frac{2}{\sqrt{29}} |1\rangle_\psi + \frac{3i}{\sqrt{29}} |0\rangle_\psi - \frac{4}{\sqrt{29}} |-1\rangle_\psi \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \langle 0 | 1 \rangle_\psi + \frac{3i}{\sqrt{29}} \langle 0 | 0 \rangle_\psi - \frac{4}{\sqrt{29}} \langle 0 | -1 \rangle_\psi \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \frac{i}{\sqrt{2}} + \frac{3i}{\sqrt{29}} 0 - \frac{4}{\sqrt{29}} \frac{-i}{\sqrt{2}} \right|^2 = \frac{1}{58} |i(2 + 4)|^2 = \frac{36}{58}\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{-1} &= |\langle -1 | \psi_{iv} \rangle|^2 = \left| \langle -1 | \left[\frac{2}{\sqrt{29}} |1\rangle_\psi + \frac{3i}{\sqrt{29}} |0\rangle_\psi - \frac{4}{\sqrt{29}} |-1\rangle_\psi \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \langle -1 | 1 \rangle_\psi + \frac{3i}{\sqrt{29}} \langle -1 | 0 \rangle_\psi - \frac{4}{\sqrt{29}} \langle -1 | -1 \rangle_\psi \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \frac{-1}{2} + \frac{3i}{\sqrt{29}} \frac{1}{\sqrt{2}} - \frac{4}{\sqrt{29}} \frac{-1}{2} \right|^2 = \frac{1}{58} |\sqrt{2} + 3i|^2 = \frac{11}{58}\end{aligned}$$

The three probabilities add to unity, as they must.

b) The possible results of a measurement of the spin component S_y are always $+1\hbar$, $0\hbar$, $-1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}\mathcal{P}_{1y} &= |\langle 1 | \psi_{iv} \rangle|^2 = \left| \langle 1 | \left[\frac{2}{\sqrt{29}} |1\rangle_\psi + \frac{3i}{\sqrt{29}} |0\rangle_\psi - \frac{4}{\sqrt{29}} |-1\rangle_\psi \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \langle 1 | 1 \rangle_\psi + \frac{3i}{\sqrt{29}} \langle 1 | 0 \rangle_\psi - \frac{4}{\sqrt{29}} \langle 1 | -1 \rangle_\psi \right|^2 = \left| \frac{2}{\sqrt{29}} \right|^2 = \frac{4}{29}\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{0y} &= |\langle 0 | \psi_{iv} \rangle|^2 = \left| \langle 0 | \left[\frac{2}{\sqrt{29}} |1\rangle_\psi + \frac{3i}{\sqrt{29}} |0\rangle_\psi - \frac{4}{\sqrt{29}} |-1\rangle_\psi \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} \langle 0 | 1 \rangle_\psi + \frac{3i}{\sqrt{29}} \langle 0 | 0 \rangle_\psi - \frac{4}{\sqrt{29}} \langle 0 | -1 \rangle_\psi \right|^2 = \left| \frac{3i}{\sqrt{29}} \right|^2 = \frac{9}{29}\end{aligned}$$

$$\begin{aligned} \mathcal{P}_{-1y} &= \left| {}_y\langle -1 | \psi_{in} \rangle \right|^2 = \left| {}_y\langle -1 | \left[\frac{2}{\sqrt{29}} |1\rangle_y + \frac{3i}{\sqrt{29}} |0\rangle_y - \frac{4}{\sqrt{29}} |-1\rangle_y \right] \right|^2 \\ &= \left| \frac{2}{\sqrt{29}} {}_y\langle -1 | 1 \rangle_y + \frac{3i}{\sqrt{29}} {}_y\langle -1 | 0 \rangle_y - \frac{4}{\sqrt{29}} {}_y\langle -1 | -1 \rangle_y \right|^2 = \left| -\frac{4}{\sqrt{29}} \right|^2 = \frac{16}{29} \end{aligned}$$

The three probabilities add to unity, as they must.

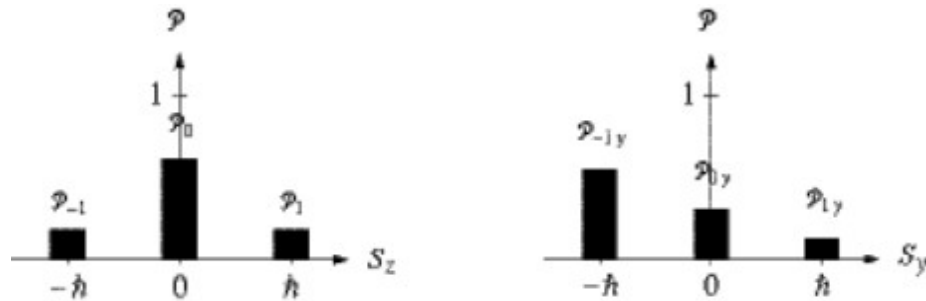
c) For the first measurement, the expectation value is

$$\langle S_z \rangle = \sum_{\mu} \mu \eta \Pi_{\mu} = 1\eta \frac{11}{58} + 0\eta \frac{36}{58} + (-1)\eta \frac{11}{58} = 0\eta$$

For the second measurement, the expectation value is

$$\langle S_y \rangle = \sum_{\mu} \mu \eta \Pi_{\mu} = 1\eta \frac{4}{29} + 0\eta \frac{9}{29} + (-1)\eta \frac{16}{29} = -\eta \frac{12}{29}$$

The histograms are shown below.



2.17 a) The possible results of a measurement of the spin component S_z are always $+1\hbar$, $0\hbar$, $-1\hbar$ for a spin-1 particle. The probabilities are

$$\mathcal{P}_{\hbar} = \left| \langle 1 | \psi \rangle \right|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{30}} 1 \right|^2 = \frac{1}{30}$$

$$\mathcal{P}_0 = \left| \langle 0 | \psi \rangle \right|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{30}} 2 \right|^2 = \frac{4}{30}$$

$$\mathcal{P}_{-\hbar} = \left| \langle -1 | \psi \rangle \right|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{30}} 5i \right|^2 = \frac{25}{30}$$

The expectation value of S_z is

$$\langle S_z \rangle = \Pi_{\hbar} \eta + \Pi_0 0 + \Pi_{-\hbar} (-\eta) = \frac{1}{30} \eta + \frac{4}{30} 0 + \frac{25}{30} (-\eta) = -\frac{24}{30} \eta = -\frac{4}{5} \eta$$

b) The expectation value of S_x is

$$\begin{aligned}
\langle S_x \rangle &= \langle \psi | \Sigma_x | \psi \rangle = \frac{1}{\sqrt{30}} \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix} \\
&= \frac{1}{30} \frac{\eta}{\sqrt{2}} \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \begin{pmatrix} 2 \\ 1+5i \\ 2 \end{pmatrix} = \frac{1}{30} \frac{\eta}{\sqrt{2}} (2 + 2(1+5i) - 5i \cdot 2) = \frac{\sqrt{2}}{15} \eta
\end{aligned}$$

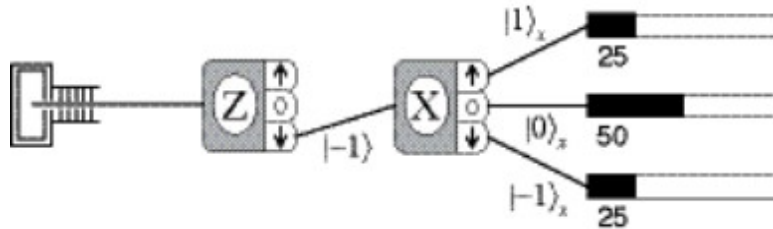
2.18 a) The possible results of a measurement of the spin component S_z are always $+1\hbar$, $0\hbar$, $-1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}
\mathcal{P}_1 &= |\langle 1 | \psi_{in} \rangle|^2 = \left| \langle 1 | \left[\frac{1}{\sqrt{14}} |1\rangle - \frac{3}{\sqrt{14}} |0\rangle + \frac{2i}{\sqrt{14}} |-1\rangle \right] \right|^2 \\
&= \left| \frac{1}{\sqrt{14}} \langle 1|1\rangle - \frac{3}{\sqrt{14}} \langle 1|0\rangle + \frac{2i}{\sqrt{14}} \langle 1|-1\rangle \right|^2 = \left| \frac{1}{\sqrt{14}} \right|^2 = \frac{1}{14} \\
\mathcal{P}_0 &= |\langle 0 | \psi_{in} \rangle|^2 = \left| \langle 0 | \left[\frac{1}{\sqrt{14}} |1\rangle - \frac{3}{\sqrt{14}} |0\rangle + \frac{2i}{\sqrt{14}} |-1\rangle \right] \right|^2 \\
&= \left| \frac{1}{\sqrt{14}} \langle 0|1\rangle - \frac{3}{\sqrt{14}} \langle 0|0\rangle + \frac{2i}{\sqrt{14}} \langle 0|-1\rangle \right|^2 = \left| -\frac{3}{\sqrt{14}} \right|^2 = \frac{9}{14} \\
\mathcal{P}_{-1} &= |\langle -1 | \psi_{in} \rangle|^2 = \left| \langle -1 | \left[\frac{1}{\sqrt{14}} |1\rangle - \frac{3}{\sqrt{14}} |0\rangle + \frac{2i}{\sqrt{14}} |-1\rangle \right] \right|^2 \\
&= \left| \frac{1}{\sqrt{14}} \langle -1|1\rangle - \frac{3}{\sqrt{14}} \langle -1|0\rangle + \frac{2i}{\sqrt{14}} \langle -1|-1\rangle \right|^2 = \left| \frac{2i}{\sqrt{14}} \right|^2 = \frac{4}{14}
\end{aligned}$$

b) After the S_z measurement, the system is in the state $|-1\rangle$. The possible results of a measurement of the spin component S_x are always $+1\hbar$, $0\hbar$, $-1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}
\mathcal{P}_{1x} &= |\langle 1 | \psi_{in} \rangle|^2 = \left| \left(\frac{1}{2} \langle 1 | + \frac{1}{\sqrt{2}} \langle 0 | + \frac{1}{2} \langle -1 | \right) |-1\rangle \right|^2 = \left| \frac{1}{2} \right|^2 = \frac{1}{4} \\
\mathcal{P}_{0x} &= |\langle 0 | \psi_{in} \rangle|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle 1 | - \frac{1}{\sqrt{2}} \langle -1 | \right) |-1\rangle \right|^2 = \left| \frac{-1}{\sqrt{2}} \right|^2 = \frac{1}{2} \\
\mathcal{P}_{-1x} &= |\langle -1 | \psi_{in} \rangle|^2 = \left| \left(\frac{1}{2} \langle 1 | - \frac{1}{\sqrt{2}} \langle 0 | + \frac{1}{2} \langle -1 | \right) |-1\rangle \right|^2 = \left| \frac{1}{2} \right|^2 = \frac{1}{4}
\end{aligned}$$

c) Schematic of experiment.



2.19 The probability is

$$\begin{aligned}
 \mathcal{P}_{\psi_f} &= \left| \langle \psi_f | \psi_i \rangle \right|^2 = \left| \left(\frac{1-i}{\sqrt{7}} \langle 1| + \frac{2}{\sqrt{7}} \langle 0| + \frac{i}{\sqrt{7}} \langle -1| \right) \left(\frac{1}{\sqrt{6}} |1\rangle - \frac{\sqrt{2}}{\sqrt{6}} |0\rangle + \frac{i\sqrt{3}}{\sqrt{6}} |-1\rangle \right) \right|^2 \\
 &= \left| \frac{1-i}{\sqrt{7}} \frac{1}{\sqrt{6}} \langle 1|1\rangle - \frac{1-i}{\sqrt{7}} \frac{\sqrt{2}}{\sqrt{6}} \langle 1|0\rangle + \frac{1-i}{\sqrt{7}} \frac{i\sqrt{3}}{\sqrt{6}} \langle 1|-1\rangle + \frac{2}{\sqrt{7}} \frac{1}{\sqrt{6}} \langle 0|1\rangle - \frac{2}{\sqrt{7}} \frac{\sqrt{2}}{\sqrt{6}} \langle 0|0\rangle + \frac{2}{\sqrt{7}} \frac{i\sqrt{3}}{\sqrt{6}} \langle 0|-1\rangle + \right. \\
 &\quad \left. + \frac{i}{\sqrt{7}} \frac{1}{\sqrt{6}} \langle -1|1\rangle - \frac{i}{\sqrt{7}} \frac{\sqrt{2}}{\sqrt{6}} \langle -1|0\rangle + \frac{i}{\sqrt{7}} \frac{i\sqrt{3}}{\sqrt{6}} \langle -1|-1\rangle \right|^2 \\
 &= \left| \frac{1-i}{\sqrt{7}} \frac{1}{\sqrt{6}} \frac{1}{2} + \frac{1-i}{\sqrt{7}} \frac{\sqrt{2}}{\sqrt{6}} \frac{i}{\sqrt{2}} - \frac{1-i}{\sqrt{7}} \frac{i\sqrt{3}}{\sqrt{6}} \frac{1}{2} + \frac{2}{\sqrt{7}} \frac{1}{\sqrt{6}} \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{7}} \frac{\sqrt{2}}{\sqrt{6}} 0 + \frac{2}{\sqrt{7}} \frac{i\sqrt{3}}{\sqrt{6}} \frac{1}{\sqrt{2}} + \right. \\
 &\quad \left. + \frac{i}{\sqrt{7}} \frac{1}{\sqrt{6}} \frac{1}{2} - \frac{i}{\sqrt{7}} \frac{\sqrt{2}}{\sqrt{6}} \frac{i}{\sqrt{2}} - \frac{i}{\sqrt{7}} \frac{i\sqrt{3}}{\sqrt{6}} \frac{1}{2} \right|^2 \\
 &= \frac{1}{168} \left| 1 - i + 2 + 2i - \sqrt{3} - i\sqrt{3} + 2\sqrt{2} - 0 + i2\sqrt{6} + i + 2 + \sqrt{3} \right|^2 \\
 &= \frac{1}{168} \left| 5 + 2\sqrt{2} + 2i - i\sqrt{3} + i2\sqrt{6} \right|^2 = \frac{1}{168} \{ (5 + 2\sqrt{2})^2 + (2 - \sqrt{3} + 2\sqrt{6})^2 \} \\
 &= \frac{1}{168} (64 + 8\sqrt{2} - 4\sqrt{3} + 8\sqrt{6}) \cong 0.524
 \end{aligned}$$

or in matrix notation

$$\begin{aligned}
 \mathcal{P}_{\psi_f} &= \left| \langle \psi_f | \psi_i \rangle \right|^2 = \left| \left\{ \frac{1-i}{\sqrt{7}} \begin{pmatrix} \frac{1}{2} & \frac{-i}{\sqrt{2}} & \frac{-1}{2} \end{pmatrix} + \frac{2}{\sqrt{7}} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} + \frac{i}{\sqrt{7}} \begin{pmatrix} \frac{1}{2} & \frac{i}{\sqrt{2}} & \frac{-1}{2} \end{pmatrix} \right\} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -\sqrt{2} \\ i\sqrt{3} \end{pmatrix} \right|^2 \\
 &= \frac{1}{168} \left| \begin{pmatrix} 1 + 2\sqrt{2} & -2\sqrt{2} - i\sqrt{2} & -1 + 2\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{2} \\ i\sqrt{3} \end{pmatrix} \right|^2 \\
 &= \frac{1}{168} \left| 1 + 2\sqrt{2} + 4 + 2i - i\sqrt{3} + i2\sqrt{6} \right|^2 = \frac{1}{168} (64 + 8\sqrt{2} - 4\sqrt{3} + 8\sqrt{6}) \cong 0.524
 \end{aligned}$$

2.20 Spin 1 unknowns. Follow the solution method given in the lab handout. (i) For unknown number 1, the measured probabilities are

$$\begin{aligned}
 \mathcal{P}_1 &= \frac{1}{4} & \Pi_{\xi} &= \frac{1}{4} & \Pi_{\psi} &= 1 \\
 \Pi_0 &= \frac{1}{2} & \Pi_{\xi} &= \frac{1}{2} & \Pi_{\psi} &= 0 \\
 \Pi_1 &= \frac{1}{4} & \Pi_{\xi} &= \frac{1}{4} & \Pi_{\psi} &= 0
 \end{aligned}$$

Write the unknown state as



Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_1 &= |\langle 1 | \psi_1 \rangle|^2 = |\langle 1 | \{ \alpha |1\rangle + \beta |0\rangle + \chi |-1\rangle \}|^2 = |\alpha|^2 = \frac{1}{4} \Rightarrow \alpha = \frac{1}{2} \\ \Pi_0 &= |\langle 0 | \psi_1 \rangle|^2 = |\langle 0 | \{ \alpha |1\rangle + \beta |0\rangle + \chi |-1\rangle \}|^2 = |\beta|^2 = \frac{1}{2} \Rightarrow \beta = \frac{1}{\sqrt{2}} \varepsilon^\alpha \\ \Pi_{-1} &= |\langle -1 | \psi_1 \rangle|^2 = |\langle -1 | \{ \alpha |1\rangle + \beta |0\rangle + \chi |-1\rangle \}|^2 = |\chi|^2 = \frac{1}{4} \Rightarrow \chi = \frac{1}{4} \varepsilon^\beta\end{aligned}$$

allowing for possible relative phases. So now the unknown state is

$$|\psi_1\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \frac{1}{2}e^{i\beta}|-1\rangle$$

Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_{0x} &= |\langle 0 | \psi_1 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \{ \langle 1 | - \langle -1 | \} \{ \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}\varepsilon^\alpha|0\rangle + \frac{1}{2}\varepsilon^\beta|-1\rangle \} \right|^2 = \left| \frac{1}{2\sqrt{2}} \{ 1 - \varepsilon^\beta \} \right|^2 \\ &= \frac{1}{8} \{ 1 - \varepsilon^\beta \} \{ 1 - \varepsilon^{-\beta} \} = \frac{1}{8} \{ 1 + 1 - \varepsilon^\beta - \varepsilon^{-\beta} \} = \frac{1}{4} \{ 1 - \chi \cos \beta \} = \frac{1}{2} \Rightarrow \chi \cos \beta = -1 \Rightarrow \beta = \pi\end{aligned}$$

Giving the state

$$|\psi_1\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|0\rangle - \frac{1}{2}|-1\rangle$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_{1y} &= |\langle 1 | \psi_1 \rangle|^2 = \left| \left\{ \frac{1}{2}\langle 1 | - \frac{1}{\sqrt{2}}\langle 0 | - \frac{1}{2}\langle -1 | \right\} \left\{ \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}\varepsilon^\alpha|0\rangle - \frac{1}{2}|-1\rangle \right\} \right|^2 = \left| \frac{1}{4} - \frac{1}{2}\varepsilon^\alpha + \frac{1}{4} \right|^2 \\ &= \frac{1}{4} \{ 1 - i\varepsilon^\alpha \} \{ 1 + i\varepsilon^{-\alpha} \} = \frac{1}{4} \{ 1 + 1 - i\varepsilon^\alpha + i\varepsilon^{-\alpha} \} = \frac{1}{2} \{ 1 + \sin \alpha \} = 1 \Rightarrow \sin \alpha = 1 \Rightarrow \alpha = \frac{\pi}{2}\end{aligned}$$

Hence the unknown state is

$$|\psi_1\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}}|0\rangle - \frac{1}{2}|-1\rangle = \frac{1}{2}|1\rangle + \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle = |1\rangle_y$$

(ii) For unknown number 2, the measured probabilities are

$$\begin{aligned}\mathcal{P}_1 &= \frac{1}{4} & \Pi_\xi &= \frac{9}{16} & \Pi_\psi &= 0.870 \\ \Pi_0 &= \frac{1}{2} & \Pi_{0\xi} &= \frac{3}{8} & \Pi_{0\psi} &= 0.125 \\ \Pi_{-1} &= \frac{1}{4} & \Pi_{-1\xi} &= \frac{1}{16} & \Pi_{-1\psi} &= 0.005\end{aligned}$$

Write the unknown state as



Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_1 &= |\langle 1 | \psi_2 \rangle|^2 = |\langle 1 | \{ \alpha |1\rangle + \beta |0\rangle + \chi |-1\rangle \}|^2 = |\alpha|^2 = \frac{1}{4} \Rightarrow \alpha = \frac{1}{2} \\ \Pi_0 &= |\langle 0 | \psi_2 \rangle|^2 = |\langle 0 | \{ \alpha |1\rangle + \beta |0\rangle + \chi |-1\rangle \}|^2 = |\beta|^2 = \frac{1}{2} \Rightarrow \beta = \frac{1}{\sqrt{2}} \varepsilon^\alpha \\ \Pi_{-1} &= |\langle -1 | \psi_2 \rangle|^2 = |\langle -1 | \{ \alpha |1\rangle + \beta |0\rangle + \chi |-1\rangle \}|^2 = |\chi|^2 = \frac{1}{4} \Rightarrow \chi = \frac{1}{2} \varepsilon^\beta\end{aligned}$$

allowing for possible relative phases. So now the unknown state is

$$|\psi_2\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \frac{1}{2}e^{i\beta}|-1\rangle$$

Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned} \mathcal{P}_{0x} &= \left| \langle 0 | \psi_2 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \{ \langle 1 | - \langle -1 | \} \{ \frac{1}{2} | 1 \rangle + \frac{1}{\sqrt{2}} e^{i\alpha} | 0 \rangle + \frac{1}{2} e^{i\beta} | -1 \rangle \} \right|^2 = \left| \frac{1}{2\sqrt{2}} \{ 1 - e^{i\beta} \} \right|^2 \\ &= \frac{1}{8} \{ 1 - e^{i\beta} \} \{ 1 - e^{-i\beta} \} = \frac{1}{8} \{ 1 + 1 - e^{i\beta} - e^{-i\beta} \} = \frac{1}{4} \{ 1 - \chi_{0\sigma\beta} \} = \frac{3}{8} \\ &\Rightarrow \chi_{0\sigma\beta} = -\frac{1}{2} \Rightarrow \beta = \frac{2\pi}{3}, \frac{4\pi}{3} \end{aligned}$$

$$\begin{aligned} \mathcal{P}_{1x} &= \left| \langle 1 | \psi_2 \rangle \right|^2 = \left| \{ \frac{1}{2} \langle 1 | + \frac{1}{\sqrt{2}} \langle 0 | + \frac{1}{2} \langle -1 | \} \{ \frac{1}{2} | 1 \rangle + \frac{1}{\sqrt{2}} e^{i\alpha} | 0 \rangle + \frac{1}{2} e^{i\beta} | -1 \rangle \} \right|^2 = \left| \frac{1}{4} + \frac{1}{2} e^{i\alpha} + \frac{1}{4} e^{i\beta} \right|^2 \\ &= \frac{1}{16} \{ 1 + 2 e^{i\alpha} + e^{i\beta} \} \{ 1 + 2 e^{-i\alpha} + e^{-i\beta} \} = \frac{1}{16} \{ 6 + 4 \chi_{0\sigma\alpha} + 2 \chi_{0\sigma\beta} + 4 \chi_{0\sigma}(\alpha - \beta) \} \\ &= \frac{1}{16} \{ 6 + 4 \chi_{0\sigma\alpha} + 2 \chi_{0\sigma\beta} + 4 \chi_{0\sigma\alpha} \chi_{0\sigma\beta} + 4 \alpha \nu \alpha \nu \beta \} \\ &= \frac{1}{16} \{ 5 + 2 \chi_{0\sigma\alpha} + 4 \alpha \nu \alpha \nu \beta \} = \frac{9}{16} \end{aligned}$$

which yields

$$\begin{aligned} \cos \alpha + 2 \alpha \nu \alpha \nu \beta &= 2 \\ 2 \alpha \nu \alpha \nu \beta &= 2 - \chi_{0\sigma\alpha} \\ 4 \alpha \nu^2 \alpha \nu^2 \beta &= 4 - 4 \chi_{0\sigma\alpha} + \chi_{0\sigma}^2 \alpha \\ 4(1 - \chi_{0\sigma}^2 \alpha) &= 4 - 4 \chi_{0\sigma\alpha} + \chi_{0\sigma}^2 \alpha \\ 4 \chi_{0\sigma}^2 \alpha - 4 \chi_{0\sigma\alpha} + 1 &= 0 \\ 2 \chi_{0\sigma\alpha} - 1 &= 0 \\ \chi_{0\sigma\alpha} &= \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned} \mathcal{P}_{1y} &= \left| \langle 1 | \psi_2 \rangle \right|^2 = \left| \{ \frac{1}{2} \langle 1 | - \frac{1}{\sqrt{2}} \langle 0 | - \frac{1}{2} \langle -1 | \} \{ \frac{1}{2} | 1 \rangle + \frac{1}{\sqrt{2}} e^{i\alpha} | 0 \rangle + \frac{1}{2} e^{i\beta} | -1 \rangle \} \right|^2 = \left| \frac{1}{4} - \frac{1}{2} e^{i\alpha} - \frac{1}{4} e^{i\beta} \right|^2 \\ &= \frac{1}{16} \{ 1 - 2 i e^{i\alpha} - e^{i\beta} \} \{ 1 + 2 i e^{-i\alpha} - e^{-i\beta} \} = \frac{1}{16} \{ 6 + 4 \alpha \nu \alpha - 2 \chi_{0\sigma\beta} - 4 \alpha \nu (\alpha - \beta) \} \\ &= \frac{1}{16} \{ 6 + 4 \alpha \nu \alpha - 2 \chi_{0\sigma\beta} - 4 \alpha \nu \alpha \chi_{0\sigma\beta} + 4 \chi_{0\sigma\alpha} \alpha \nu \beta \} \\ &= \frac{1}{16} \{ 7 + 6 \alpha \nu \alpha + 4 \chi_{0\sigma\alpha} \alpha \nu \beta \} = 0.87 \end{aligned}$$

which gives

$$\begin{aligned}
 3\sin\alpha + 2\cos\alpha \sin\beta &= 3.46 \approx 2\sqrt{3} \\
 2\cos\alpha \sin\beta &= 2\sqrt{3} - 3\sin\alpha \\
 4\cos^2\alpha \sin^2\beta &= 12 - 12\sqrt{3}\sin\alpha + 9\sin^2\alpha \\
 4(1 - \sin^2\alpha) \frac{3}{4} &= 12 - 12\sqrt{3}\sin\alpha + 9\sin^2\alpha \\
 \sin^2\alpha - \sqrt{3}\sin\alpha + \frac{3}{4} &= 0 \\
 \sin\alpha - \frac{\sqrt{3}}{2} &= 0 \\
 \sin\alpha &= \frac{\sqrt{3}}{2} \Rightarrow \alpha = \frac{\pi}{3}, \frac{2\pi}{3} \Rightarrow \alpha = \frac{\pi}{3} \Rightarrow \beta = \frac{2\pi}{3}
 \end{aligned}$$

Hence the unknown state is

$$|\psi_2\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}e^{i\frac{\pi}{3}}|0\rangle + \frac{1}{2}e^{i\frac{2\pi}{3}}|-1\rangle = |1\rangle_{n(\theta=\frac{\pi}{2}, \phi=\frac{\pi}{3})}$$

(iii) For unknown number 3, the measured probabilities are

$$\begin{aligned}
 \mathcal{P}_1 &= \frac{1}{3} & \Pi_{\xi} &= \frac{1}{6} & \Pi_{\psi} &= 0.0286 \\
 \Pi_0 &= \frac{1}{3} & \Pi_{\xi} &= \frac{2}{3} & \Pi_{\psi} &= 0 \\
 \Pi_{-1} &= \frac{1}{3} & \Pi_{\xi} &= \frac{1}{6} & \Pi_{\psi} &= 0.9714
 \end{aligned}$$

Write the unknown state as



Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned}
 \mathcal{P}_1 &= |\langle 1|\psi_3\rangle|^2 = |\langle 1|\{\alpha|1\rangle + \beta|0\rangle + \chi|-1\rangle\}|^2 = |\alpha|^2 = \frac{1}{3} \Rightarrow \alpha = \frac{1}{\sqrt{3}} \\
 \Pi_0 &= |\langle 0|\psi_3\rangle|^2 = |\langle 0|\{\alpha|1\rangle + \beta|0\rangle + \chi|-1\rangle\}|^2 = |\beta|^2 = \frac{1}{3} \Rightarrow \beta = \frac{1}{\sqrt{3}}e^{i\alpha} \\
 \Pi_{-1} &= |\langle -1|\psi_3\rangle|^2 = |\langle -1|\{\alpha|1\rangle + \beta|0\rangle + \chi|-1\rangle\}|^2 = |\chi|^2 = \frac{1}{3} \Rightarrow \chi = \frac{1}{\sqrt{3}}e^{i\beta}
 \end{aligned}$$

allowing for possible relative phases. So now the unknown state is

$$|\psi_3\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}e^{i\alpha}|0\rangle + \frac{1}{\sqrt{3}}e^{i\beta}|-1\rangle$$

Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned}
 \mathcal{P}_{0x} &= |\langle 0|\psi_3\rangle|^2 = \left| \frac{1}{\sqrt{2}}\{\langle 1| - \langle -1|\} \frac{1}{\sqrt{3}}\{|1\rangle + e^{i\alpha}|0\rangle + e^{i\beta}|-1\rangle\} \right|^2 = \left| \frac{1}{\sqrt{6}}\{1 - e^{i\beta}\} \right|^2 \\
 &= \frac{1}{6}\{1 - e^{i\beta}\}\{1 - e^{-i\beta}\} = \frac{1}{6}\{1 + 1 - e^{i\beta} - e^{-i\beta}\} = \frac{1}{3}\{1 - \cos\beta\} = \frac{2}{3} \\
 &\Rightarrow \cos\beta = -1 \Rightarrow \beta = \pi
 \end{aligned}$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_{1y} &= \left| \langle 1 | \psi_3 \rangle \right|^2 = \left| \left\{ \frac{1}{2} \langle 1 | - \frac{i}{\sqrt{2}} \langle 0 | - \frac{1}{2} \langle -1 | \right\} \frac{1}{\sqrt{3}} \{ |1\rangle + \epsilon^\alpha |0\rangle - |-1\rangle \} \right|^2 = \left| \frac{1}{\sqrt{3}} \left(\frac{1}{2} - \frac{i}{\sqrt{2}} \epsilon^\alpha + \frac{1}{2} \right) \right|^2 \\ &= \frac{1}{3} \left\{ 1 - \frac{i}{\sqrt{2}} \epsilon^\alpha \right\} \left\{ 1 + \frac{i}{\sqrt{2}} \epsilon^{-\alpha} \right\} = \frac{1}{3} \left\{ \frac{3}{2} + \sqrt{2} \alpha \nu \alpha \right\} = 0.0286 \\ &\Rightarrow \alpha \nu \alpha = -1 \Rightarrow \alpha = \frac{3\pi}{2}\end{aligned}$$

Hence the unknown state is

$$|\psi_3\rangle = \frac{1}{\sqrt{3}}|1\rangle - \frac{i}{\sqrt{3}}|0\rangle - \frac{1}{\sqrt{3}}|-1\rangle \quad \{ \neq |m\rangle_n \}$$

(iv) For unknown number 4, the measured probabilities are

$$\begin{aligned}\mathcal{P}_1 &= \frac{1}{2} & \Pi_{1\xi} &= \frac{1}{4} & \Pi_{1\psi} &= \frac{1}{4} \\ \Pi_0 &= 0 & \Pi_{0\xi} &= \frac{1}{2} & \Pi_{0\psi} &= \frac{1}{2} \\ \Pi_{-1} &= \frac{1}{2} & \Pi_{-1\xi} &= \frac{1}{4} & \Pi_{-1\psi} &= \frac{1}{4}\end{aligned}$$

Write the unknown state as



Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_1 &= \left| \langle 1 | \psi_4 \rangle \right|^2 = \left| \langle 1 | \{ \alpha |1\rangle + \beta |0\rangle + \chi |-1\rangle \} \right|^2 = |\alpha|^2 = \frac{1}{2} \Rightarrow \alpha = \frac{1}{\sqrt{2}} \\ \Pi_0 &= \left| \langle 0 | \psi_4 \rangle \right|^2 = \left| \langle 0 | \{ \alpha |1\rangle + \beta |0\rangle + \chi |-1\rangle \} \right|^2 = |\beta|^2 = 0 \Rightarrow \beta = 0 \\ \Pi_{-1} &= \left| \langle -1 | \psi_4 \rangle \right|^2 = \left| \langle -1 | \{ \alpha |1\rangle + \beta |0\rangle + \chi |-1\rangle \} \right|^2 = |\chi|^2 = \frac{1}{2} \Rightarrow \chi = \frac{1}{\sqrt{2}} \epsilon^{i\beta}\end{aligned}$$

allowing for possible relative phases. So now the unknown state is

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}e^{i\beta}|-1\rangle$$

Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned}\mathcal{P}_{0x} &= \left| \langle 0 | \psi_4 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \{ \langle 1 | - \langle -1 | \} \frac{1}{\sqrt{2}} \{ |1\rangle + \epsilon^{i\beta} |-1\rangle \} \right|^2 = \left| \frac{1}{2} \{ 1 - \epsilon^{i\beta} \} \right|^2 \\ &= \frac{1}{4} \{ 1 - \epsilon^{i\beta} \} \{ 1 - \epsilon^{-i\beta} \} = \frac{1}{4} \{ 1 + 1 - \epsilon^{i\beta} - \epsilon^{-i\beta} \} = \frac{1}{2} \{ 1 - \chi \circ \sigma \beta \} = \frac{1}{2} \\ &\Rightarrow \chi \circ \sigma \beta = 0 \Rightarrow \beta = \frac{\pi}{2}, \frac{3\pi}{2}\end{aligned}$$

Giving

$$|\psi_4\rangle = \frac{1}{\sqrt{2}} \{ |1\rangle \pm i |-1\rangle \}$$

with no more info from S_x , S_y , S_z measurements. In the SPINS program choose **n** at angles $\theta = 90^\circ$, $\phi = 45^\circ$, 225° to see that the unknown state is

$$|\psi_4\rangle = \frac{1}{\sqrt{2}} \{ |1\rangle - i |-1\rangle \} = |0\rangle_{n(\theta=\frac{\pi}{2}, \phi=\frac{\pi}{4})} = |0\rangle_{n(\theta=\frac{\pi}{2}, \phi=\frac{5\pi}{4})}$$

2.21. The spin-1 interferometer had an S_z SG device, an S_x device, and an S_z SG device. The S_z eigenstates are $|1\rangle, |0\rangle, |-1\rangle$. The S_x eigenstates are

$$\begin{aligned} |1\rangle_x &= \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle \\ |0\rangle_x &= \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|-1\rangle \\ |-1\rangle_x &= \frac{1}{2}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle \end{aligned} .$$

Let $|\psi_i\rangle$ be the quantum state after the i^{th} Stern-Gerlach device. The first SG device transmit particles with $S_z = +\eta$, so the state $|\psi_1\rangle$ is

$$|\psi_1\rangle = |1\rangle.$$

The second SG device transmits particles in 1, 2, or 3 of the S_x eigenstates $|1\rangle_x, |0\rangle_x, |-1\rangle_x$. To find $|\psi_2\rangle$, we use the projection postulate:

$$|\psi_2\rangle = \frac{P_n |\psi_1\rangle}{\sqrt{\langle \psi_1 | P_n | \psi_1 \rangle}}$$

where P_n is the projection operator onto the measured states. For example, if the second SG device transmits particles with $S_x = +\eta$, we get

$$|\psi_2\rangle = \frac{P_{1x} |\psi_1\rangle}{\sqrt{\langle \psi_1 | P_{1x} | \psi_1 \rangle}} = \frac{|1\rangle_x \langle 1|1\rangle}{\sqrt{\langle 1|1\rangle_x \langle 1|1\rangle}} = |1\rangle_x$$

as expected. In matrix notation, the S_x eigenstates are

$$|1\rangle_x \text{ B } \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} \quad |0\rangle_x \text{ B } \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \quad |-1\rangle_x \text{ B } \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

and the projection operators are

$$\begin{aligned}
P_{1x} &= |1\rangle_{\xi} \langle 1| B \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix} \\
P_{0\xi} &= |0\rangle_{\xi} \langle 0| B \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \\
P_{-1\xi} &= |-1\rangle_{\xi} \langle -1| B \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix}
\end{aligned}$$

The probability of measuring a result after the third SG device is $\| \langle \psi_2 | \psi_3 \rangle \|^2$. We want to calculate the three probabilities

$$\begin{aligned}
P_1 &= |\langle 1 | \psi_2 \rangle|^2 \\
P_0 &= |\langle 0 | \psi_2 \rangle|^2 \\
P_{-1} &= |\langle -1 | \psi_2 \rangle|^2
\end{aligned}$$

for all possible (7) cases of 1 beam, 2 beams, or 3 beams from SG2.

(i) When the second SG device transmits particles with $S_x = +1\hbar$ only:

$$\begin{aligned}
|\psi_2\rangle &= \frac{P_{1x}|\psi_1\rangle}{\sqrt{\langle \psi_1 | P_{1x} | \psi_1 \rangle}} = \frac{P_{1x}|1\rangle}{\sqrt{\langle 1 | P_{1x} | 1 \rangle}} \\
P_{1x}|1\rangle B \begin{pmatrix} \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{4} \end{pmatrix} \\
\langle 1 | P_{1x} | 1 \rangle B \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{4} \end{pmatrix} &= 1/4 \\
|\psi_2\rangle B \frac{1}{\sqrt{1/4}} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{4} \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}
\end{aligned}$$

which is $|1\rangle_x$ as expected. The three probabilities are

$$\begin{aligned}\mathcal{P}_1 &= |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \frac{1}{4} \\ \Pi_0 &= |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \frac{1}{2} \\ \Pi_1 &= |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \frac{1}{4}\end{aligned}$$

(ii) When the second SG device transmits particles with $S_x = 0\hbar$, $|\psi_2\rangle = |0\rangle_x$ and the three probabilities are

$$\begin{aligned}\mathcal{P}_1 &= |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2 = \frac{1}{2} \\ \Pi_0 &= |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2 = 0 \\ \Pi_1 &= |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2 = \frac{1}{2}\end{aligned}$$

(iii) When the second SG device transmits particles with $S_x = -1\hbar$, $|\psi_2\rangle = |-1\rangle_x$ and the three probabilities are

$$P_1 = |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \frac{1}{4}$$

$$P_0 = |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \frac{1}{2}$$

$$P_{-1} = |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \frac{1}{4}$$

(iv) When the second SG device transmits particles with $S_x = +\eta$ **and** $S_x = 0\eta$ in a coherent beam

$$|\psi_2\rangle = \frac{(P_{1x} + P_{0x})|\psi_1\rangle}{\sqrt{\langle \psi_1 | (P_{1x} + P_{0x}) | \psi_1 \rangle}} = \frac{(P_{1x} + P_{0x})|1\rangle}{\sqrt{\langle 1 | (P_{1x} + P_{0x}) | 1 \rangle}}$$

$$P_{0x}|1\rangle B \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}$$

$$(P_{1x} + P_{0x})|1\rangle B \begin{pmatrix} 1/4 \\ 1/2\sqrt{2} \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/2\sqrt{2} \\ -1/4 \end{pmatrix}$$

$$\langle 1 | (P_{1x} + P_{0x}) | 1 \rangle B \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3/4 \\ 1/2\sqrt{2} \\ -1/4 \end{pmatrix} = 3/4$$

$$|\psi_2\rangle B \frac{1}{\sqrt{3/4}} \begin{pmatrix} 3/4 \\ 1/2\sqrt{2} \\ -1/4 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix}$$

The three probabilities are

$$\begin{aligned} P_1 &= |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \right|^2 = \frac{3}{4} \\ P_0 &= |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \right|^2 = \frac{1}{6} \\ P_{-1} &= |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \right|^2 = \frac{1}{12} \end{aligned}$$

(v) When the second SG device transmits particles with $S_x = +\eta$ **and** $S_x = -1\eta$ in a coherent beam

$$\begin{aligned} |\psi_2\rangle &= \frac{(P_{1x} + P_{-1x})|\psi_1\rangle}{\sqrt{\langle \psi_1 | (P_{1x} + P_{-1x}) | \psi_1 \rangle}} = \frac{(P_{1x} + P_{-1x})|1\rangle}{\sqrt{\langle 1 | (P_{1x} + P_{-1x}) | 1 \rangle}} \\ P_{-1x}|1\rangle &= \begin{pmatrix} \frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} \\ (P_{1x} + P_{-1x})|1\rangle &= \begin{pmatrix} 1/4 \\ 1/2\sqrt{2} \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \\ \langle 1 | (P_{1x} + P_{-1x}) | 1 \rangle &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} = 1/2 \\ |\psi_2\rangle &= \frac{1}{\sqrt{1/2}} \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \end{aligned}$$

The three probabilities are

$$\begin{aligned} \mathcal{P}_1 &= |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right|^2 = \frac{1}{2} \\ \Pi_0 &= |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right|^2 = 0 \\ \Pi_{-1} &= |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right|^2 = \frac{1}{2} \end{aligned}$$

(vi) When the second SG device transmits particles with $S_x = 0\hbar$ **and** $S_x = -1\hbar$ in a coherent beam

$$\begin{aligned} |\psi_2\rangle &= \frac{(P_{0x} + P_{-1x})|\psi_1\rangle}{\sqrt{\langle \psi_1 | (P_{0x} + P_{-1x}) | \psi_1 \rangle}} = \frac{(P_{0x} + P_{-1x})|1\rangle}{\sqrt{\langle 1 | (P_{0x} + P_{-1x}) | 1 \rangle}} \\ (P_{0x} + P_{-1x})|1\rangle &= \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} = \begin{pmatrix} 3/4 \\ -1/2\sqrt{2} \\ -1/4 \end{pmatrix} \\ \langle 1 | (P_{0x} + P_{-1x}) | 1 \rangle &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3/4 \\ -1/2\sqrt{2} \\ -1/4 \end{pmatrix} = 3/4 \\ |\psi_2\rangle &= \frac{1}{\sqrt{3/4}} \begin{pmatrix} 3/4 \\ -1/2\sqrt{2} \\ -1/4 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ -1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \end{aligned}$$

The three probabilities are

$$\begin{aligned} P_1 &= |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ -1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \right|^2 = \frac{3}{4} \\ P_0 &= |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ -1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \right|^2 = \frac{1}{6} \\ P_{-1} &= |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ -1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix} \right|^2 = \frac{1}{12} \end{aligned}$$

(vii) When the second SG device transmits particles with $S_x = +\eta$, $S_x = 0\eta$, and $S_x = -1\eta$ in a coherent beam.

$$\begin{aligned} |\psi_2\rangle &= \frac{(P_{1x} + P_{0x} + P_{-1x})|\psi_1\rangle}{\sqrt{\langle \psi_1 | (P_{1x} + P_{0x} + P_{-1x}) | \psi_1 \rangle}} = \frac{(P_{1x} + P_{0x} + P_{-1x})|1\rangle}{\sqrt{\langle 1 | (P_{1x} + P_{0x} + P_{-1x}) | 1 \rangle}} \\ P_{-1x}|1\rangle &= \begin{pmatrix} 1/4 & -1/2\sqrt{2} & 1/4 \\ -1/2\sqrt{2} & 1/2 & -1/2\sqrt{2} \\ 1/4 & -1/2\sqrt{2} & 1/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} \\ (P_{1x} + P_{0x} + P_{-1x})|1\rangle &= \begin{pmatrix} 1/4 \\ 1/2\sqrt{2} \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \langle 1 | (P_{1x} + P_{0x} + P_{-1x}) | 1 \rangle &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \\ |\psi_2\rangle &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

which is $|1\rangle$ as expected. The three probabilities are

$$\mathcal{P}_1 = |\langle 1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right|^2 = 1$$

$$\mathcal{P}_0 = |\langle 0 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right|^2 = 0$$

$$\mathcal{P}_{-1} = |\langle -1 | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = 0$$

The cumulated results are

$ \psi_2\rangle$	$\mathcal{P}_1$	$\mathcal{P}_0$	$\mathcal{P}_{-1}$
$ 1\rangle_x$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$ 0\rangle_x$	$\frac{1}{2}$	0	$\frac{1}{2}$
$ -1\rangle_x$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$ 1\rangle_x \& 0\rangle_x$	$\frac{3}{4}$	$\frac{1}{6}$	$\frac{1}{12}$
$ 1\rangle_x \& -1\rangle_x$	$\frac{1}{2}$	0	$\frac{1}{2}$
$ 0\rangle_x \& -1\rangle_x$	$\frac{3}{4}$	$\frac{1}{6}$	$\frac{1}{12}$
$ 1\rangle_x \& 0\rangle_x \& -1\rangle_x$	1	0	0

2.22 a) The probability of measuring spin up at the 2nd Stern-Gerlach analyzer and spin down at the 3rd Stern-Gerlach analyzer is the product of the individual probabilities:

$$\mathcal{P}_{+\rightarrow+n\rightarrow-z} = \mathcal{P}_{+\rightarrow+n} \mathcal{P}_{+n\rightarrow-z} = |\langle + | + \rangle_n|^2 |\langle - | + \rangle_n|^2$$

The $|+\rangle_n$ eigenstate is

$$|+\rangle_n = \cos\frac{\theta}{2}|+\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-\rangle$$

so the probability is

$$\begin{aligned} \mathcal{P}_{+\rightarrow+n\rightarrow-z} &= \left| \langle \cos\frac{\theta}{2}\langle + | + e^{-i\varphi}\sin\frac{\theta}{2}\langle - | \right| + \rangle^2 \left| \langle - | (\cos\frac{\theta}{2}|+\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-\rangle) \right|^2 \\ &= \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} = \frac{1}{4}\sin^2\theta \end{aligned}$$

b) To maximize the probability requires that $\theta = \pi/2$, and the probability is



c) If the 2nd Stern-Gerlach analyzer is removed, then the probability is

$$\mathcal{P}_{+ \rightarrow -z} = |\langle - | + \rangle|^2 = 0$$

because the two states are orthogonal.

2.23 (a) The commutator is

$$\begin{aligned}
 [A, B] &= AB - B A B \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & 0 & \beta_2 \\ 0 & \beta_2 & 0 \end{pmatrix} - \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & 0 & \beta_2 \\ 0 & \beta_2 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \\
 &= B \begin{pmatrix} \alpha_1 \beta_1 & 0 & 0 \\ 0 & 0 & \alpha_2 \beta_2 \\ 0 & \alpha_3 \beta_2 & 0 \end{pmatrix} - \begin{pmatrix} \alpha_1 \beta_1 & 0 & 0 \\ 0 & 0 & \alpha_3 \beta_2 \\ 0 & \alpha_2 \beta_2 & 0 \end{pmatrix} \\
 &= B \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta_2 (\alpha_2 - \alpha_3) \\ 0 & \beta_2 (\alpha_3 - \alpha_2) & 0 \end{pmatrix} \neq 0
 \end{aligned}$$

so they do not commute.

(b) A is already diagonal, so the eigenvalues and eigenvectors are obtained by inspection. The eigenvalues are

$$a_1, a_2, a_3$$

and the eigenvectors are

$$|a_1\rangle = |1\rangle B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |a_2\rangle = |2\rangle B \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |a_3\rangle = |3\rangle B \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For B , diagonalization yields the eigenvalues

$$\begin{aligned}
 \begin{pmatrix} b_1 - \lambda & 0 & 0 \\ 0 & -\lambda & \beta_2 \\ 0 & \beta_2 & -\lambda \end{pmatrix} = 0 &\Rightarrow (\beta_1 - \lambda)(\lambda^2 - \beta_2^2) = 0 \\
 \Rightarrow \lambda &= \beta_1, \beta_2, -\beta_2
 \end{aligned}$$

and the eigenvectors

$$\begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} \nu \\ \varpi \\ \omega \end{pmatrix} = \beta_1 \begin{pmatrix} \nu \\ \varpi \\ \omega \end{pmatrix} \Rightarrow \begin{aligned} \beta_1 \nu &= \beta_1 \nu \\ \beta_2 \omega &= \beta_1 \varpi \Rightarrow \omega = \varpi = 0 \\ \beta_2 \varpi &= \beta_1 \omega \end{aligned}$$

$$|\nu|^2 + |\varpi|^2 + |\omega|^2 = 1 \Rightarrow |\nu|^2 = 1 \Rightarrow \nu = 1, \varpi = 0, \omega = 0 \Rightarrow |\beta_1\rangle = |1\rangle B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} \nu \\ \varpi \\ \omega \end{pmatrix} = \beta_2 \begin{pmatrix} \nu \\ \varpi \\ \omega \end{pmatrix} \Rightarrow \begin{aligned} \beta_1 \nu &= \beta_2 \nu \\ \beta_2 \omega &= \beta_2 \varpi \Rightarrow \nu = 0, \omega = \varpi \\ \beta_2 \varpi &= \beta_2 \omega \end{aligned}$$

$$\langle \beta_2 | \beta_2 \rangle = 1 \Rightarrow |\varpi|^2 + |\omega|^2 = 1 \Rightarrow \nu = 0, \varpi = \frac{1}{\sqrt{2}}, \omega = \frac{1}{\sqrt{2}} \Rightarrow |\beta_2\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) B \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} \nu \\ \varpi \\ \omega \end{pmatrix} = -\beta_2 \begin{pmatrix} \nu \\ \varpi \\ \omega \end{pmatrix} \Rightarrow \begin{aligned} \beta_1 \nu &= -\beta_2 \nu \\ \beta_2 \omega &= -\beta_2 \varpi \Rightarrow \nu = 0, \omega = -\varpi \\ \beta_2 \varpi &= -\beta_2 \omega \end{aligned}$$

$$\langle -\beta_2 | -\beta_2 \rangle = 1 \Rightarrow |\varpi|^2 + |\omega|^2 = 1 \Rightarrow \nu = 0, \varpi = \frac{1}{\sqrt{2}}, \omega = -\frac{1}{\sqrt{2}} \Rightarrow |-\beta_2\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) B \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

c) If B is measured, the possible results are the allowed eigenvalues $b_1, b_2, -\beta_2$. If the initial state is $|\psi_i\rangle = |2\rangle$, then the probabilities are

$$\mathcal{P}_{b_1} = |\langle \beta_1 | \psi_i \rangle|^2 = |\langle 1 | 2 \rangle|^2 = 0$$

$$\mathcal{P}_{b_2} = |\langle \beta_2 | \psi_i \rangle|^2 = \left| \frac{1}{\sqrt{2}} \langle 2 | + \langle 3 | \rangle | 2 \rangle \right|^2 = \frac{1}{2}$$

$$\mathcal{P}_{-\beta_2} = |\langle -\beta_2 | \psi_i \rangle|^2 = \left| \frac{1}{\sqrt{2}} \langle 2 | - \langle 3 | \rangle | 2 \rangle \right|^2 = \frac{1}{2}$$

If A is then measured, the possible results are the allowed eigenvalues a_1, a_2, a_3 . If b_2 was the first result, then the new state is $|b_2\rangle$ and when A is measured the subsequent probabilities are

$$\mathcal{P}_{a_1} = |\langle \alpha_1 | \beta_2 \rangle|^2 = \left| \langle 1 | \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) \rangle \right|^2 = 0$$

$$\mathcal{P}_{a_2} = |\langle \alpha_2 | \beta_2 \rangle|^2 = \left| \langle 2 | \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) \rangle \right|^2 = \frac{1}{2}$$

$$\mathcal{P}_{a_3} = |\langle \alpha_3 | \beta_2 \rangle|^2 = \left| \langle 3 | \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) \rangle \right|^2 = \frac{1}{2}$$

If $-b_2$ was the first result, then the new state is $|-b_2\rangle$ and when A is measured the subsequent probabilities are

$$\mathcal{P}_{a_1} = |\langle \alpha_1 | -\beta_2 \rangle|^2 = \left| \langle 1 | \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) \rangle \right|^2 = 0$$

$$\mathcal{P}_{a_2} = |\langle \alpha_2 | -\beta_2 \rangle|^2 = \left| \langle 2 | \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) \rangle \right|^2 = \frac{1}{2}$$

$$\mathcal{P}_{a_3} = |\langle \alpha_3 | -\beta_2 \rangle|^2 = \left| \langle 3 | \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) \rangle \right|^2 = \frac{1}{2}$$

d) If two operators do not commute, then the corresponding observables cannot be measured simultaneously. Part (a) tells us that the operators A and B not commute. Part (c) tells us that measurement B "disturbs" the measurement of A so the two measurements are not compatible (cannot be made simultaneously).

2.24 (a) The eigenvalue equations for the S_z operator and the four eigenstates are

$$S_z |+\frac{3}{2}\rangle = +\frac{3}{2}\hbar |+\frac{3}{2}\rangle$$

$$S_z |+\frac{1}{2}\rangle = +\frac{1}{2}\hbar |+\frac{1}{2}\rangle$$

$$S_z |-\frac{1}{2}\rangle = -\frac{1}{2}\hbar |-\frac{1}{2}\rangle$$

$$S_z |-\frac{3}{2}\rangle = -\frac{3}{2}\hbar |-\frac{3}{2}\rangle$$

(b) The matrix representations of the S_z eigenstates are the unit vectors

$$|+\frac{3}{2}\rangle \text{ B } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |+\frac{1}{2}\rangle \text{ B } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |-\frac{1}{2}\rangle \text{ B } \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |-\frac{3}{2}\rangle \text{ B } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(c) The matrix representation of the S_z operator has the eigenvalues along the diagonal:

$$S_z \text{ B } \begin{pmatrix} +\frac{3}{2}\hbar & 0 & 0 & 0 \\ 0 & +\frac{1}{2}\hbar & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\hbar & 0 \\ 0 & 0 & 0 & -\frac{3}{2}\hbar \end{pmatrix}$$

(d) The eigenvalue equations for the S^2 operator follow from the general equation $S^2 |s m_s\rangle = \alpha(\alpha+1)\hbar^2 |s m_s\rangle$

$$\mathbf{S}^2 \left| +\frac{3}{2} \right\rangle = \frac{15}{4} \hbar^2 \left| +\frac{3}{2} \right\rangle$$

$$\mathbf{S}^2 \left| +\frac{1}{2} \right\rangle = \frac{15}{4} \hbar^2 \left| +\frac{1}{2} \right\rangle$$

$$\mathbf{S}^2 \left| -\frac{1}{2} \right\rangle = \frac{15}{4} \hbar^2 \left| -\frac{1}{2} \right\rangle$$

$$\mathbf{S}^2 \left| -\frac{3}{2} \right\rangle = \frac{15}{4} \hbar^2 \left| -\frac{3}{2} \right\rangle$$

where we have suppressed the s label.

(e) The matrix representation of the $\mathbf{S}^2$ operator has the eigenvalues along the diagonal:

$$\mathbf{S}^2 \text{ B } \begin{pmatrix} \frac{15}{4} \hbar^2 & 0 & 0 & 0 \\ 0 & \frac{15}{4} \hbar^2 & 0 & 0 \\ 0 & 0 & \frac{15}{4} \hbar^2 & 0 \\ 0 & 0 & 0 & \frac{15}{4} \hbar^2 \end{pmatrix}$$

2.25 The projection operators P_+ and P_- are represented by the matrices



The Hermitian adjoints of these matrices are obtained by transposing and complex conjugating them, yielding

$$P_+^\dagger \text{ B } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_-^\dagger \text{ B } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Since the Hermitian adjoints are equal to the original matrices, these operators are Hermitian.
