

Contents

0.1 Solutions Ch. 1	iii
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1.

(i) The orthonormality of the states is demonstrated as follows

$\langle \alpha_1 | \alpha_1 \rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1; \langle \alpha_1 | \alpha_2 \rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$. Similarly one can show $\langle \alpha_2 | \alpha_1 \rangle = 0$ and $\langle \alpha_2 | \alpha_2 \rangle = 1$

(ii) The column matrix can be written as

$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(iii) The outer products $|\alpha_i\rangle \langle \alpha_j|$ give the following matrices

$$\begin{aligned} |\alpha_1\rangle \langle \alpha_1| &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; |\alpha_1\rangle \langle \alpha_2| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \\ |\alpha_2\rangle \langle \alpha_1| &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; |\alpha_2\rangle \langle \alpha_2| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

(iv) The $|\alpha_i\rangle'$ s satisfy completeness relation from the following relation

$$\sum_i |\alpha_i\rangle \langle \alpha_i| = |\alpha_1\rangle \langle \alpha_1| + |\alpha_2\rangle \langle \alpha_2| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{1}$$

(v) write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = a |\alpha_1\rangle \langle \alpha_1| + b |\alpha_1\rangle \langle \alpha_2| + c |\alpha_2\rangle \langle \alpha_1| + d |\alpha_2\rangle \langle \alpha_2|$$

(vi)

$$A |\alpha_1\rangle = + |\alpha_1\rangle \text{ and } A |\alpha_2\rangle = - |\alpha_2\rangle$$

Constructing the matrix elements from the above relation and using orthonormality we find

$$\{A\} = \begin{bmatrix} \langle \alpha_1 | A | \alpha_1 \rangle & \langle \alpha_1 | A | \alpha_2 \rangle \\ \langle \alpha_2 | A | \alpha_1 \rangle & \langle \alpha_2 | A | \alpha_2 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

2. Start with the relation

$$AA^{-1} = 1$$

Take the derivative with respect to λ

$$\frac{d}{d\lambda} (AA^{-1}) = 0, \text{ therefore}$$

$A \frac{dA^{-1}}{d\lambda} + \frac{dA}{d\lambda} A^{-1} = 0$, multiplying on the left by A^{-1} and then moving the second term to the left gives

$$\frac{dA^{-1}}{d\lambda} = -A^{-1} \frac{dA}{d\lambda} A^{-1}$$

3.

For an operator A ,

$$AA^{-1} = 1, \text{ therefore } (AA^{-1})^\dagger = 1 \text{ or } (A^{-1})^\dagger = (A^\dagger)^{-1}$$

$$U = \frac{\mathbf{1} + iK}{\mathbf{1} - iK} = (\mathbf{1} + iK)(\mathbf{1} - iK)^{-1} = (\mathbf{1} - iK)^{-1}(\mathbf{1} + iK)$$

the last step follows from the fact that K 's commute among themselves

Therefore,

$$U^\dagger = (\mathbf{1} - iK)^\dagger \left((\mathbf{1} + iK)^{-1} \right)^\dagger = (\mathbf{1} - iK)(\mathbf{1} + iK)^{-1} \text{ since } K \text{ is Hermitian, and}$$

$$\text{and } UU^\dagger = (\mathbf{1} + iK) \left[(\mathbf{1} - iK)^{-1} (\mathbf{1} - iK) \right] (\mathbf{1} + iK)^{-1} = (\mathbf{1} + iK)(\mathbf{1} + iK)^{-1} =$$

1

One can write

$$e^{iC} = \frac{e^{iC/2}}{e^{-iC/2}} = \frac{1 + i \tan C/2}{1 - i \tan C/2}$$

and identify

$$K = \tan C/2$$

One can also show that

$$U = e^{iC} = \cos C + i \sin C$$

If $U = A + iB$ then identifying

$A = \cos C$, $B = \sin C$ we note that A and B commute.

4.

Let U be a unitary operator diagonalizing A , so that

$$A_D = UAU^\dagger$$

is a diagonal matrix. Then

$$\text{Tr}(A) = \text{Tr}(A_D)$$

$$\det(A) = \det(A_D)$$

Similarly by expanding e^A in powers of A we get

$$\det(e^A) = \det(e^{A_D}) = e^{(A_D)_{11}} \cdot e^{(A_D)_{22}} \cdot e^{(A_D)_{33}} \cdot \dots = e^{\text{Tr}(A_D)} = e^{\text{Tr}(A)}$$

5.

$$\text{Tr}[|\alpha\rangle\langle\beta|] = \sum_n \langle n|\alpha\rangle\langle\beta|n\rangle = \sum_n \langle\beta|n\rangle\langle n|\alpha\rangle = \langle\beta|\alpha\rangle$$

6.

$$A = |\alpha\rangle\langle\alpha| + \lambda|\beta\rangle\langle\alpha| + \lambda^*|\alpha\rangle\langle\beta| + \mu|\beta\rangle\langle\beta|$$

In the matrix form it can be written (take $|\alpha\rangle = |1\rangle$, $|\beta\rangle = |2\rangle$ to implement matrix notation)

$$\{A\} = \begin{bmatrix} 1 & \lambda^* \\ \lambda & \mu \end{bmatrix}$$

Let a be the eigenvalues, then

$$(1-a)(\mu-a) - |\lambda|^2 = 0$$

The solutions are

$$a = \frac{(1+\mu) \pm \sqrt{(1-\mu)^2 + 4|\lambda|^2}}{2}$$

$$(i) \lambda = 1, \mu = +1$$

$$a = \frac{2 \pm \sqrt{4|\lambda|^2}}{2} = 1 \pm |\lambda|^2$$

$$\lambda = 1, \mu = -1$$

$$a = \pm\sqrt{1+|\lambda|^2}$$

$$(ii) \lambda = i, \mu = +1$$

$$a = 2, 0$$

$$\lambda = i, \mu = -1$$

$$a = \pm\sqrt{2}$$