SOLUTIONS TO CHAPTER 2
Background

2.1 The DFT of a sequence $x(n)$ of length N may be expressed in matrix form as follows

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

where $\mathbf{x} = [x(0),\ x(1),\ \ldots, x(N-1)]^T$ is a vector containing the signal values and $\mathbf{X}$ is a vector containing the DFT coefficients $X(k)$,

(a) Find the matrix $\mathbf{W}$.

(b) What properties does the matrix $\mathbf{W}$ have?

(c) What is the inverse of $\mathbf{W}$?

Solution

(a) The DFT of a sequence $x(n)$ of length N is

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} x(n)W_N^{nk}$$

where $W_N \equiv e^{-j\frac{2\pi}{N}}$. If we define

$$\mathbf{w}_k^H = \left[1, W_N^k, W_N^{2k}, \ldots, W_N^{k(N-1)}\right]$$

then $X(k)$ is the inner product

$$X(k) = \mathbf{w}_k^H \cdot \mathbf{x}$$

Arranging the DFT coefficients in a vector we have,

$$\mathbf{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} \mathbf{w}_0^H\mathbf{x} \\ \mathbf{w}_1^H\mathbf{x} \\ \vdots \\ \mathbf{w}_{N-1}^H\mathbf{x} \end{bmatrix} = \mathbf{W}\mathbf{x}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_0^H \\ \mathbf{w}_1^H \\ \vdots \\ \mathbf{w}_{N-1}^H \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

(b) The matrix $\mathbf{W}$ is *symmetric* and nonsingular. In addition, due to the orthogonality of the complex exponentials,

$$\mathbf{w}_k^H \cdot \mathbf{w}_l = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-l)} = \begin{cases} N & ; \quad k = l \\ 0 & ; \quad k \neq l \end{cases}$$

it follows that $\mathbf{W}$ is *orthogonal.*

(c) Due to the orthogonality of $\mathbf{W}$, the inverse is

$$\mathbf{W}^{-1} = \tfrac{1}{N}\mathbf{W}^H$$

2.2 Prove or disprove each of the following statements

(a) The product of two upper triangular matrices is upper triangular.

(b) The product of two Toeplitz matrices is Toeplitz.

(c) The product of two centrosymmetric matrices is centrosymmetric.

Solution

(a) With

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

it follows that if $\mathbf{A}$ is upper triangular then $a_{ij} = 0$ for all $i < j$. If $\mathbf{B}$ is also upper triangular, then the (i,j)th element of the product $\mathbf{C} = \mathbf{AB}$ is

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

For $i < j$ we have

$$c_{ij} = \sum_{k=1}^{j-1} a_{ik} \cdot b_{kj} + \sum_{k=j}^{n} a_{ik} \cdot b_{kj}$$

The first summation is equal to zero since $b_{kj} = 0$ for $k = 1, \ldots, j-1$, and the second term is equal to zero since $a_{ik} = 0$ for $k = j, \ldots, n$. Therefore, $c_{ij} = 0$ for $i < j$ and $\mathbf{C}$ is upper triangular.

(b) The product of two Toeplitz matrices is *not* necessarily Toeplitz. This may be easily demonstrated by example. Let $\mathbf{A}$ be the following 3×3 Toeplitz matrix,

$$\mathbf{A} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} \\ a_1 & a_0 & a_{-1} \\ a_2 & a_1 & a_0 \end{bmatrix}$$

and let $\mathbf{B}$ be the Toeplitz matrix

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The product, $\mathbf{AB}$, is

$$\mathbf{AB} = \begin{bmatrix} a_{-2} & 0 & a_0 \\ a_{-1} & 0 & a_1 \\ a_0 & 0 & a_2 \end{bmatrix}$$

which is *not* Toeplitz.

(c) If $\mathbf{A}$ and $\mathbf{B}$ are centrosymmetric matrices, then

$$\mathbf{A} = \mathbf{J}^H\mathbf{A}\mathbf{J} \quad ; \quad \mathbf{B} = \mathbf{J}^H\mathbf{B}\mathbf{J}$$

and

$$\mathbf{AB} = (\mathbf{J}^H\mathbf{A}\mathbf{J})(\mathbf{J}^H\mathbf{B}\mathbf{J})$$

Since $\mathbf{J}\mathbf{J}^H = \mathbf{I}$, then

$$\mathbf{AB} = \mathbf{J}^H \mathbf{ABJ}$$

which means that $\mathbf{AB}$ is centrosymmetric.

2.3 Find the minimum norm solution to the following set of underdetermined linear equations,

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution

With

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

since the rows of $\mathbf{A}$ are linearly independent, then the minimum norm solution is unique and given by

$$\mathbf{x}_0 = \mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1}\mathbf{b}$$

With

$$\mathbf{A}\mathbf{A}^H = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$$

and

$$(\mathbf{A}\mathbf{A}^H)^{-1} = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

it follows that the minimum norm solution is

$$\mathbf{x} = \frac{1}{14}\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -3 \\ 8 \\ 10 \\ 3 \end{bmatrix}$$

2.4 Consider the set of inconsistent linear equations $\mathbf{Ax} = \mathbf{b}$ given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(a) Find the least squares solution to these equations.
(b) Find the projection matrix $\mathbf{P}_A$.
(c) Find the best approximation $\widehat{\mathbf{b}} = \mathbf{P}_A\mathbf{b}$ to $\mathbf{b}$.
(d) Consider the matrix

$$\mathbf{P}_{\mathbf{A}}^{\perp} = \mathbf{I} - \mathbf{P}_A$$

Find the vector $\mathbf{b}^{\perp} = \mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{b}$ and show that it is orthogonal to $\widehat{\mathbf{b}}$. What does the matrix $\mathbf{P}_A^{\perp}$ represent?

Solution

(a) Since the columns of $\mathbf{A}$ are linearly independent, the least squares solution is unique and given by

$$\mathbf{x}_0 = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{b}$$

With

$$\mathbf{A}^H\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

it follows that

$$(\mathbf{A}^H\mathbf{A})^{-1} = \tfrac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

and, therefore,

$$\begin{aligned} \mathbf{x}_0 &= \tfrac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \tfrac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \tfrac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

(b) The projection matrix is

$$\begin{aligned} \mathbf{P}_A &= \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H = \tfrac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \tfrac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \tfrac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \end{aligned}$$

(c) The best approximation to $\mathbf{b}$ is

$$\widehat{\mathbf{b}} = \mathbf{P}_A\mathbf{b} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(d) The matrix $\mathbf{P}_{\mathbf{A}}^{\perp}$ is

$$\mathbf{P}_{\mathbf{A}}^{\perp} = \mathbf{I} - \mathbf{P}_A = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

and

$$\mathbf{b}^{\perp} = \mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{b} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

The inner product between $\widehat{\mathbf{b}}$ and $\mathbf{b}^{\perp}$ is

$$< \widehat{\mathbf{b}},\ \mathbf{b}^{\perp} >= \frac{1}{9}\begin{bmatrix} 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = 0$$

Therefore, $\widehat{\mathbf{b}}$ is *orthogonal* to $\mathbf{b}^{\perp}$. The matrix $\mathbf{P}_{\mathbf{A}}^{\perp}$ is a projection matrix that projects a vector onto the space that is orthogonal to the space spanned by the columns of $\mathbf{A}$.

2.5 Consider the problem of trying to model a sequence $x(n)$ as the sum of a constant plus a complex exponential of frequency ω_0,

$$\widehat{x}(n) = c + ae^{jn\omega_0} \quad ; \quad n = 0, 1, \ldots, N-1$$

where c and a are unknown. We may express the problem of finding the values for c and a as one of solving a set of overdetermined linear equations

$$\begin{bmatrix} 1 & 1 \\ 1 & e^{j\omega_0} \\ \vdots & \vdots \\ 1 & e^{j(N-1)\omega_0} \end{bmatrix} \begin{bmatrix} c \\ a \end{bmatrix} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

(a) Find the least squares solution for c and a.

(b) If N is even and $\omega_0 = 2\pi k/N$ for some integer k, find the least squares solution for c and a.

Solution

(a) Assuming that $\omega_0 \neq 0, 2\pi, \ldots$, the columns of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & e^{j\omega_0} \\ \vdots & \vdots \\ 1 & e^{j(N-1)\omega_0} \end{bmatrix}$$

are linearly independent, and the least squares solution for c and a is given by

$$\begin{bmatrix} c \\ a \end{bmatrix} = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{x}$$

Since

$$\mathbf{A}^H\mathbf{A} = \begin{bmatrix} N & \displaystyle\sum_{n=0}^{N-1} e^{jn\omega_0} \\ \displaystyle\sum_{n=0}^{N-1} e^{-jn\omega_0} & N \end{bmatrix} = \begin{bmatrix} N & \dfrac{1-e^{jN\omega_0}}{1-e^{j\omega_0}} \\ \dfrac{1-e^{-jN\omega_0}}{1-e^{-j\omega_0}} & N \end{bmatrix}$$

Therefore, the inverse of $(\mathbf{A}^H\mathbf{A})$ is

$$(\mathbf{A}^H\mathbf{A})^{-1} = \frac{1}{N^2 - \dfrac{1-\cos N\omega_0}{1-\cos\omega_0}} \begin{bmatrix} N & -\dfrac{1-e^{jN\omega_0}}{1-e^{j\omega_0}} \\ -\dfrac{1-e^{-jN\omega_0}}{1-e^{-j\omega_0}} & N \end{bmatrix}$$

and we have

$$\begin{bmatrix} c \\ a \end{bmatrix} = \frac{1}{N^2 - \dfrac{1-\cos N\omega_0}{1-\cos\omega_0}} \begin{bmatrix} N & -\dfrac{1-e^{jN\omega_0}}{1-e^{j\omega_0}} \\ -\dfrac{1-e^{-jN\omega_0}}{1-e^{-j\omega_0}} & N \end{bmatrix} \begin{bmatrix} \displaystyle\sum_{n=0}^{N-1} x(n) \\ \displaystyle\sum_{n=0}^{N-1} x(n)e^{-jn\omega_0} \end{bmatrix}$$

which becomes

$$\begin{bmatrix} c \\ a \end{bmatrix} = \frac{1}{N^2 - \dfrac{1-\cos N\omega_0}{1-\cos\omega_0}} \begin{bmatrix} N\displaystyle\sum_{n=0}^{N-1} x(n) - \dfrac{1-e^{jN\omega_0}}{1-e^{j\omega_0}} \displaystyle\sum_{n=0}^{N-1} x(n)e^{-jn\omega_0} \\ N\displaystyle\sum_{n=0}^{N-1} x(n)e^{-jn\omega_0} - \dfrac{1-e^{-jN\omega_0}}{1-e^{-j\omega_0}} \displaystyle\sum_{n=0}^{N-1} x(n) \end{bmatrix}$$

(b) If $\omega_0 = 2\pi k/N$ and $k \neq 0$, then

$$\frac{1-e^{jN\omega_0}}{1-e^{j\omega_0}} = \frac{1-e^{-jN\omega_0}}{1-e^{-j\omega_0}} = 0$$

and

$$\frac{1-\cos N\omega_0}{1-\cos\omega_0} = 0$$

Therefore, we have

$$\begin{bmatrix} c \\ a \end{bmatrix} = \begin{bmatrix} \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} x(n) \\ \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} x(n)e^{-jn\omega_0} \end{bmatrix}$$

2.6 It is known that the sum of the squares of n from $n = 1$ to $N - 1$ has a closed form expression of the following form

$$\sum_{n=0}^{N-1} n^2 = a_0 + a_1 N + a_2 N^2 + a_3 N^3$$

Given that a third-order polynomial is uniquely determined in terms of the values of the polynomial at four distinct points, derive a closed form expression for this sum by setting up a set of linear equations and solving these equations for a_0, a_1, a_2, a_3. Compare your solution to that given in Table 2.3.

Solution

Assuming that

$$\sum_{n=0}^{N-1} n^2 = a_0 + a_1 N + a_2 N^2 + a_3 N^3$$

we may evaluate this sum for $N = 1, 2, 3, 4$ and write down the following set of four equations in four unknowns

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 14 \end{bmatrix}$$

Solving these equations for a_0, a_1, and a_2, we find

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/6 \\ -1/2 \\ 1/3 \end{bmatrix}$$

which gives the following closed-form expression for the sum,

$$\sum_{n=0}^{N-1} n^2 = \frac{1}{6}N - \frac{1}{2}N^2 + \frac{1}{3}N^3 = \frac{1}{6}N(N-1)(2N-1)$$

2.7 Show that a projection matrix $\mathbf{P}_A$ has the following two properties,

1. It is *idempotent*, $\mathbf{P}_A^2 = \mathbf{P}_A$.
2. It is Hermitian.

Solution

Given a matrix $\mathbf{A}$, the projection matrix $\mathbf{P}_A$ is

$$\mathbf{P}_A = \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$$

Therefore,

$$\begin{aligned}\mathbf{P}_A^2 &= \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H \\ &= \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H = \mathbf{P}_A\end{aligned}$$

and it follows that $\mathbf{P}_A$ is idempotent. Also,

$$\mathbf{P}_A^H = \left[\mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\right]^H = \mathbf{A}\big[(\mathbf{A}^H\mathbf{A})^{-1}\big]^H\mathbf{A}^H$$

Since $\mathbf{A}\mathbf{A}^H$ is Hermitian, then so is its inverse,

$$\big[(\mathbf{A}^H\mathbf{A})^{-1}\big]^H = (\mathbf{A}^H\mathbf{A})^{-1}$$

and

$$\mathbf{P}_A^H = \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$$

Thus, $\mathbf{P}_A$ is Hermitian.

2.8 Let $\mathbf{A} > 0$ and $\mathbf{B} > 0$ be positive definite matrices. Prove or disprove the following statements.

(a) $\mathbf{A}^2 > 0$.

(b) $\mathbf{A}^{-1} > 0$.

(c) $\mathbf{A} + \mathbf{B} > 0$.

Solution

(a) Let $\mathbf{v}_k$ be an eigenvector and λ_k the corresponding eigenvalue of $\mathbf{A}$. Since

$$\mathbf{A}^2\mathbf{v}_k = \mathbf{A}(\mathbf{A}\mathbf{v}_k) = \lambda_k\mathbf{A}\mathbf{v}_k = \lambda_k^2\mathbf{v}_k$$

then $\mathbf{v}_k$ is an eigenvector of $\mathbf{A}^2$ and λ_k^2 is the corresponding eigenvalue. If $\mathbf{A} > 0$, then $\lambda_k > 0$. Therefore, $\lambda_k^2 > 0$, and it follows that $\mathbf{A}^2 > 0$.

(b) If $\mathbf{A} > 0$, then the eigenvalues of $\mathbf{A}$ are positive, $\lambda_k > 0$. In addition, $\mathbf{A}^{-1}$ exists and the eigenvalues of $\mathbf{A}^{-1}$ are λ_k^{-1}. Since $\lambda_k > 0$, it follows that $\lambda_k^{-1} > 0$ and, therefore, $\mathbf{A}^{-1} > 0$.

(c) Let $\mathbf{v} \neq 0$ be an arbitrary vector. Then

$$\mathbf{v}^H(\mathbf{A} + \mathbf{B})\mathbf{v} = \mathbf{v}^H\mathbf{A}\mathbf{v} + \mathbf{v}^H\mathbf{B}\mathbf{v}$$

If $\mathbf{A} > 0$ and $\mathbf{B} > 0$, then

$$\mathbf{v}^H\mathbf{A}\mathbf{v} > 0 \quad ; \quad \mathbf{v}^H\mathbf{B}\mathbf{v} > 0$$

Therefore,

$$\mathbf{v}^H(\mathbf{A} + \mathbf{B})\mathbf{v} > 0$$

and it follows that $(\mathbf{A} + \mathbf{B}) > 0$.

2.9 (a) Prove that each eigenvector of a symmetric Toeplitz matrix is either symmetric or anti-symmetric, i.e., $\mathbf{v}_k = \pm \mathbf{J}\mathbf{v}_k$.

(b) What property can you state about the eigenvalues of a Hermitian Toeplitz matrix?

Solution

(a) If $\mathbf{A}$ is a symmetric Toeplitz matrix, then

$$\mathbf{J}^T \mathbf{A} \mathbf{J} = \mathbf{A}$$

where $\mathbf{J}$ is the exchange matrix. If $\mathbf{v}_k$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ_k, then

$$\mathbf{A}\mathbf{v}_k = \lambda_k \mathbf{v}_k$$

and, using the identity above, we have

$$\mathbf{J}^T \mathbf{A} \mathbf{J} \mathbf{v}_k = \lambda_k \mathbf{v}_k$$

Since $\mathbf{J}$ is unitary, $\mathbf{J}^T\mathbf{J} = \mathbf{I}$, if we multiply both sides of this equation on the left by $\mathbf{J}$, it follows that

$$\mathbf{A}\mathbf{J}\mathbf{v}_k = \lambda_k \mathbf{J}\mathbf{v}_k$$

Therefore, if $\mathbf{v}_k$ is an eigenvector with eigenvalue λ_k, then $\mathbf{J}\mathbf{v}_k$ is also an eigenvector with the same eigenvalue. Consequently, if the eigenvalue λ_k is *distinct*, then $\mathbf{v}_k$ and $\mathbf{J}\mathbf{v}_k$ must be equal to within a constant,

$$\mathbf{v}_k = c\mathbf{J}\mathbf{v}_k$$

However, since the exchange matrix reverses the order of the elements of the vector $\mathbf{v}_k$, the only possible values for this constant are $c = \pm 1$. Therefore,

$$\mathbf{v}_k = \pm \mathbf{J}\mathbf{v}_k$$

and the eigenvector $\mathbf{v}_k$ is either symmetric or anti-symmetric.

Now let us consider the case in which the eigenvalue λ_k is *not distinct.* We will assume that the multiplicity is two. The following discussion may be easily generalized to higher multiplicities. In this case, $\mathbf{v}_k$ and $\mathbf{J}\mathbf{v}_k$ span a two-dimensional space, and any two linearly independent vectors in this space may be selected as the eigenvectors. Therefore, we may choose

$$\widetilde{\mathbf{v}}_{k_1} = \mathbf{v}_k + \mathbf{J}\mathbf{v}_k$$

and

$$\widetilde{\mathbf{v}}_{k_2} = \mathbf{v}_k - \mathbf{J}\mathbf{v}_k$$

as the two eigenvectors. Note that $\widetilde{\mathbf{v}}_{k_1}$ is symmetric and $\widetilde{\mathbf{v}}_{k_2}$ is anti-symmetric. This completes the proof.

(b) In the case of Hermitian Toeplitz matrices, the eigenvectors are either Hermitian or anti-Hermitian, i.e.,

$$\mathbf{v}_k = \pm \mathbf{v}_k^*$$

2.10 (a) Find the eigenvalues and eigenvectors of the real 2×2 symmetric Toeplitz matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

(b) Find the eigenvalues and eigenvectors of the 2×2 Hermitian matrix

$$\mathbf{A} = \begin{bmatrix} a & b^* \\ b & a \end{bmatrix}$$

Solution

(a) The eigenvalues are the roots of the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = (a - \lambda)^2 - b^2 = 0$$

Expanding the quadratic in λ we have

$$\lambda^2 - 2a\lambda + (a^2 - b^2) = \big[\lambda - (a + b)\big]\big[\lambda - (a - b)\big] = 0$$

Therefore, the eigenvalues are $\lambda_1 = a + b$ and $\lambda_2 = a - b$. The eigenvectors, on the other hand, are solutions to the equation

$$\mathbf{A}\mathbf{v}_k = \lambda_k \mathbf{v}_k$$

For the first eigenvector, $\mathbf{v}_1$, we have

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = (a + b) \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$$

which gives $v_{11} = v_{12}$, or

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly, the eigenvector $\mathbf{v}_2$ is found to be

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(b) With

$$\mathbf{A} = \begin{bmatrix} a & b^* \\ b & a \end{bmatrix}$$

the eigenvalues are the roots of the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = (a - \lambda)^2 - |b|^2 = 0$$

or,

$$\lambda^2 - 2a\lambda + a^2 + |b|^2 = \big[\lambda - (a + |b|)\big]\big[\lambda - (a - |b|)\big] = 0$$

Thus, $\lambda_1 = a + |b|$ and $\lambda_2 = a - |b|$.

The eigenvector that has eigenvalue λ_1 is the solution to

$$\begin{bmatrix} a & b^* \\ b & a \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = (a + |b|) \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$$

which gives $v_{12} = \dfrac{b}{|b|} v_{11}$, or

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ b/|b| \end{bmatrix}$$

Similarly, for $\mathbf{v}_2$ we have

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -b/|b| \end{bmatrix}$$

2.11 Establish Property 5 on p. 45.

Solution

Let $\mathbf{B}$ be an $n \times n$ matrix with eigenvalues λ_k and eigenvectors $\mathbf{v}_k$. With

$$\mathbf{A} = \mathbf{B} + \alpha\mathbf{I}$$

note that

$$\begin{aligned} \mathbf{A}\mathbf{v}_k &= \mathbf{B}\mathbf{v}_k + \alpha\mathbf{v}_k \\ &= \lambda_k\mathbf{v}_k + \alpha\mathbf{v}_k = (\lambda_k + \alpha)\mathbf{v}_k \end{aligned}$$

Therefore, $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvectors, and the eigenvalues of $\mathbf{A}$ are $\lambda_i + \alpha$.

2.12 A necessary and sufficient condition for a Hermitian matrix $\mathbf{A}$ to be positive definite is that there exists a non-singular matrix $\mathbf{W}$ such that

$$\mathbf{A} = \mathbf{W}^H\mathbf{W}$$

(a) Prove this result.

(b) Find a factorization of the form $\mathbf{A} = \mathbf{W}^H\mathbf{W}$ for the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Solution

(a) If $\mathbf{A} > 0$, then $\mathbf{A}$ may be factored as follows

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$$

where $\mathbf{\Lambda} = \text{diag}\{\lambda_1, \ldots, \lambda_N\}$ with $\lambda_i > 0$. Therefore, $\mathbf{\Lambda}$ may be factored as follows

$$\mathbf{\Lambda} = \mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}$$

where $\mathbf{\Lambda}^{1/2} = \text{diag}\{\lambda_1^{1/2}, \ldots, \lambda_N^{1/2}\} > 0$. Thus, we may write

$$\mathbf{A} = \left(\mathbf{V}\mathbf{\Lambda}^{1/2}\right)\left(\mathbf{\Lambda}^{1/2}\mathbf{V}^H\right) = \left(\mathbf{\Lambda}^{1/2}\mathbf{V}^H\right)^H\left(\mathbf{\Lambda}^{1/2}\mathbf{V}^H\right) = \mathbf{W}^H\mathbf{W}$$

where $\mathbf{W} = \mathbf{\Lambda}^{1/2}\mathbf{V}^H > 0$ is nonsingular.

Conversely, suppose that $\mathbf{A}$ may be factored as

$$\mathbf{A} = \mathbf{W}^H\mathbf{W}$$

where $\mathbf{W}$ is a nonsingular matrix. Then $\mathbf{W}$ may be factored as follows

$$\mathbf{W} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$$

where $\mathbf{\Lambda}$ is a diagonal matrix and $\mathbf{V}$ is a unitary matrix. Thus,

$$\mathbf{A} = \mathbf{W}^H\mathbf{W} = \left(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^H\right)^H\left(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^H\right) = \mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^H$$

Since the diagonal terms of $\mathbf{\Lambda}^2$ are positive, then $\mathbf{A} > 0$.

(b) The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 3$ and $\lambda_2 = 1$, and the normalized eigenvectors are

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad ; \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore,

$$\mathbf{W}^H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \sqrt{3} \\ 1 & -\sqrt{3} \end{bmatrix}$$

2.13 Consider the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of **A**.

(b) Are the eigenvectors unique? Are they linearly independent? Are they orthogonal?

(c) Diagonalize **A**, i.e., find **V** and **D** such that

$$\mathbf{V}^H \mathbf{A} \mathbf{V} = \mathbf{D}$$

where **D** is a diagonal matrix.

Solution

(a) The eigenvalues are the roots of the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 + 1 = 0$$

which are $\lambda = \pm j$. The eigenvector corresponding to the eigenvalue $\lambda_1 = j$ satisfies the equation

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = j \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

which implies that $v_2 = j v_1$. Therefore, the normalized eigenvector is

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ j \end{bmatrix}$$

Similarly for the eigenvector corresponding to the eigenvalue $\lambda_2 = -j$ we have

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -j \end{bmatrix}$$

(b) The eigenvectors are unique, linearly independent, and orthogonal,

$$\langle \mathbf{v}_1,\ \mathbf{v}_2 \rangle = \mathbf{v}_1^H \mathbf{v}_2 = 0$$

(c) With **V** the matrix of normalized eigenvectors,

$$\mathbf{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$$

we have

$$\mathbf{V}^H \mathbf{A} \mathbf{V} = \mathbf{D}$$

where

$$\mathbf{D} = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}$$

2.14 Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

Solution

The eigenvalues of a matrix $\mathbf{A}$ are the roots of the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

For the given matrix, we have

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \det \begin{pmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} \\ &= (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2) \end{aligned}$$

Therefore, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$. The eigenvectors are found by solving the equations

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i \quad ; \quad i = 1, 2$$

For $\lambda_1 = 3$ we have

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 3 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$$

The first equation is

$$v_{11} - v_{12} = 3v_{11}$$

or

$$v_{12} = -2v_{11}$$

Therefore, the eigenvector is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Repeating for $\lambda_2 = 2$ we find

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2.15 Consider the following 3×3 symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

(b) Find the determinant of $\mathbf{A}$.

(c) Find the spectral decomposition of $\mathbf{A}$.

(d) What are the eigenvalues of $\mathbf{A}+\mathbf{I}$ and how are the eigenvectors related to those of $\mathbf{A}$?

Solution

(a) The eigenvalues are found from the roots of the characteristic equation,

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

The roots are $\lambda = 3, 1, 0$. Given the eigenvalues, the eigenvectors are found by solving the equations $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for $i = 1, 2, 3$. The eigenvectors (unnormalized) are

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1; \mathbf{v}_2; \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

(b) The determinant is equal to the product of the eigenvalues,

$$\det \mathbf{A} = \prod_{i=1}^{3} \lambda_i = 0$$

(c) The spectral decomposition for $\mathbf{A}$ is

$$\mathbf{A} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \mathbf{v}_i^H$$

where $\mathbf{v}_i$ are the *normalized* eigenvectors of $\mathbf{A}$. Since $\lambda_3 = 0$, this decomposition becomes

$$\begin{aligned} \mathbf{A} &= 3 \cdot \left(\tfrac{1}{6}\right) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \\ &= \tfrac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{aligned}$$

(d) If the eigenvalues of $\mathbf{A}$ are λ_i, then the eigenvalues of $\mathbf{A}+\mathbf{I}$ are $\lambda_i + 1$, and the eigenvectors are the same. Therefore, the eigenvalues of $\mathbf{A}+\mathbf{I}$ are $\lambda = 4, 2, 1$.

2.16 Suppose that an $n \times n$ matrix $\mathbf{A}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

(a) What are the eigenvalues and eigenvectors of $\mathbf{A}^2$?

(b) What are the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$?

Solution

(a) With $\mathbf{v}_i$ an eigenvector of $\mathbf{A}$ with eigenvalue λ_i, note that

$$\mathbf{A}^2\mathbf{v}_i = \mathbf{A}(\mathbf{A}\mathbf{v}_i) = \lambda_i(\mathbf{A}\mathbf{v}_i) = \lambda_i^2\mathbf{v}_i$$

Therefore, the eigenvectors of $\mathbf{A}^2$ are the same as those for $\mathbf{A}$, and the eigenvalues are λ_i^2.

(b) Since

$$\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i$$

then, assuming that $\mathbf{A}^{-1}$ exists,

$$\mathbf{v}_i = \lambda_i\mathbf{A}^{-1}\mathbf{v}_i$$

or

$$\mathbf{A}^{-1}\mathbf{v}_i = \frac{1}{\lambda_i}\mathbf{v}_i$$

Therefore, $\mathbf{A}^{-1}$ has the same eigenvectors as $\mathbf{A}$, and the eigenvalues are $1/\lambda_i$.

2.17 Find a matrix whose eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$ with eigenvectors $\mathbf{v}_1 = [3, 1]^T$ and $\mathbf{v}_2 = [2, 1]^T$.

Solution

From the given information, we have

$$\mathbf{A}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad ; \quad \mathbf{A}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

Let

$$\mathbf{A} = [\mathbf{a}_1, \ \mathbf{a}_2]$$

Then we have

$$3\mathbf{a}_1 + \mathbf{a}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and

$$2\mathbf{a}_1 + \mathbf{a}_2 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

Subtracting these two equations gives

$$\mathbf{a}_1 = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$$

Also, we have

$$\mathbf{a}_2 = \begin{bmatrix} 8 \\ 4 \end{bmatrix} - 2\mathbf{a}_1 = \begin{bmatrix} 18 \\ 10 \end{bmatrix}$$

Therefore,

$$\mathbf{A} = \begin{bmatrix} -5 & 18 \\ -3 & 10 \end{bmatrix}$$

2.18 Gerschgorin's circle theorem states that every eigenvalue of a matrix $\mathbf{A}$ lies in at least one of the circles $C_1, \ldots, C_N$ in the complex plane where C_i has center at the diagonal entry a_{ii} and its radius is $r_i = \sum_{j \neq i} |a_{ij}|$.

1. Prove this theorem by using the eigenvalue equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ to write

$$(\lambda - a_{ii})x_i = \sum_{j \neq i} a_{ij} x_j$$

and then use the triangle inequality,

$$\left| \sum_{j \neq i} a_{ij} x_j \right| \leq \sum_{j \neq i} |a_{ij} x_j|$$

2. Use this theorem to establish the bound on λ_{max} given in Property 7.
3. The matrix

$$\begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & 0 \\ 3 & 2 & 6 \end{bmatrix}$$

is said to be *diagonally dominant* since $|a_{ii}| > r_i$. Use Gerschgorin's circle theorem to show that this matrix is nonsingular.

Solution

1. Let $\mathbf{x} = \left[x_1, \ldots, x_N\right]^T$ be an eigenvector, and λ the corresponding eigenvalue for the matrix $\mathbf{A}$. Assume that x_i is the largest component of $\mathbf{x}$, i.e, $|x_i| \geq |x_j|$ for all $j \neq i$. With $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, it follows that

$$\sum_{j=1}^{N} a_{ij} x_j = \lambda x_i$$

or,

$$(\lambda - a_{ii})x_i = \sum_{j \neq i} a_{ij} x_j$$

Therefore,

$$|\lambda - a_{ii}| = \left| \sum_{j \neq i} a_{ij} \frac{x_j}{x_i} \right| \leq \sum_{j \neq i} |a_{ij}| \left| \frac{x_j}{x_i} \right|$$

Since $|x_i| \geq |x_j|$ for all $j \neq i$, then the ratios $|x_j/x_i|$ are less than or equal to one, and λ lies in the ith circle defined by

$$|\lambda_i - a_{ii}| \leq r_i$$

where

$$r_i = \sum_{j \neq i} |a_{ij}|$$

2. From Gerschgorin's circle theorem, for each eigenvalue, λ, there is an i such that

$$|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|$$

Since

$$|\lambda| - |a_{ii}| \leq |\lambda - a_{ii}|$$

then

$$|\lambda| \leq \sum_{j=1}^{n} |a_{ij}|$$

Therefore,

$$|\lambda_{\max}| \leq \max_i \sum_{j=1}^{n} |a_{ij}|$$

3. Let $\mathbf{A}$ be a matrix that is diagonally dominant,

$$|a_{ii}| > r_i$$

Assume that one of the eigenvalues is zero ($\mathbf{A}$ is singular). From Gerschgorin's circle theorem, we know that, for each eigenvalue,

$$|\lambda - a_{ii}| \leq r_i$$

However, if $\lambda_k = 0$, then

$$|\lambda_k - a_{ii}| = |a_{ii}| \leq r_i$$

for some i. Therefore, $\mathbf{A}$ is not diagonally dominant, which contradicts the hypothesis. Thus, if $\mathbf{A}$ is diagonally dominant, then it cannot have any zero eigenvalues and must, therefore, be nonsingular.

2.19 Consider the following quadratic function of two variables z_1 and z_2,

$$f(z_1, z_2) = 3z_1^2 + 3z_2^2 + 4z_1 z_2 + 8$$

Find the values of z_1 and z_2 that minimize $f(z_1, z_2)$ subject to the constraint that $z_1 + z_2 = 1$ and determine the minimum value of $f(z_1, z_2)$.

Solution

To minimize the function

$$f(z_1, z_2) = 3z_1^2 + 3z_2^2 + 4z_1 z_2 + 8$$

subject to the constraint

$$z_1 + z_2 = 1$$

we may use Lagrange multipliers as follows. If we define the objective function $Q(z_1, z_2)$ as follows

$$Q(z_1, z_2) = 3z_1^2 + 3z_2^2 + 4z_1 z_2 + 8 + \lambda(1 - z_1 - z_2)$$

then the values for z_1 and z_2 that minimize $f(z_1, z_2)$ may be found by solving the equations

$$\begin{aligned}
\frac{\partial}{\partial z_1} Q(z_1, z_2) &= 6z_1 + 4z_2 - \lambda = 0 \\
\frac{\partial}{\partial z_2} Q(z_1, z_2) &= 6z_2 + 4z_1 - \lambda = 0 \\
\frac{\partial}{\partial \lambda} Q(z_1, z_2) &= 1 - z_1 - z_2 = 0
\end{aligned}$$

Writing the first two equations in matrix form we have

$$\begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for z_1 and z_2 we find

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{\lambda}{20} \begin{bmatrix} 6 & -4 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\lambda}{10} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Plugging these values into the third equation above, we may solve for the Lagrange multiplier, λ, as follows,

$$1 - z_1 - z_2 = 1 - \tfrac{\lambda}{10} - \tfrac{\lambda}{10} = 1 - \tfrac{\lambda}{5} = 0$$

or

$$\lambda = 5$$

Given λ we may explicitly evaluate z_1 and z_2,

$$z_1 = 1/2 \quad ; \quad z_2 = 1/2$$

Substituting these values into $f(z_1, z_2)$ we find that the minimum value of the function is

$$\min\big[f(z_1, z_2)\big] = 10.5$$

SOLUTIONS TO CHAPTER 3
Discrete Time Random Processes

3.1 Let x be a random variable with mean m_x and variance σ_x^2. Let x_i for $i = 1, 2, \ldots, N$ be N independent measurements of the random variable x.

(a) With $\widehat{m}_x$ the sample mean defined by

$$\widehat{m}_x = \frac{1}{N}\sum_{i=1}^{N} x_i$$

determine whether or not the sample variance

$$\widehat{\sigma}_x^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \widehat{m}_x)^2$$

is unbiased, i.e., is $E\{\widehat{\sigma}_x^2\} = \sigma_x^2$?

(b) If x is a Gaussian random variable, find the variance of the sample variance, $E\{(\widehat{\sigma}_x^2 - E\{\widehat{\sigma_x^2}\})^2\}$.

Solution

(a) The expected value of the sample variance is

$$E\{\widehat{\sigma}_x^2\} = E\left\{\frac{1}{N}\sum_{i=1}^{N}\Big(x_i - \frac{1}{N}\sum_{j=1}^{N}x_j\Big)^2\right\} = \frac{1}{N}\sum_{i=1}^{N}E\Big\{\Big[(x_i - m_x) - \frac{1}{N}\sum_{j=1}^{N}(x_j - m_x)\Big]^2\Big\}$$

Expanding the square we have

$$E\{\widehat{\sigma}_x^2\} = \frac{1}{N}\sum_{i=1}^{N}E\left\{(x_i - m_x)^2 - \frac{2}{N}\sum_{j=1}^{N}(x_i - m_x)(x_j - m_x) + \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i - m_x)(x_j - m_x)\right\}$$

Since the measurements are assumed to be independent, then

$$E\big\{(x_i - m_x)(x_j - m_x)\big\} = \begin{cases} \sigma_x^2 & ; \quad i = j \\ 0 & ; \quad i \neq j \end{cases}$$

and the expression for $\widehat{\sigma}_x^2$ becomes

$$E\{\widehat{\sigma}_x^2\} = \frac{1}{N}\sum_{i=1}^{N}\Big\{\sigma_x^2 - \frac{2}{N}\sigma_x^2 + \frac{1}{N^2}N\sigma_x^2\Big\} = \sigma_x^2\big(1 - \frac{1}{N}\big) = \sigma_x^2\frac{N-1}{N}$$

Therefore, although the sample variance is biased, it is asymptotically unbiased.

(b) Finding the variance of the sample variance directly is very tedious. A simpler way is as follows. With

$$\widehat{\sigma}_x^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \widehat{m}_x)^2$$

it is well-known that

$$\frac{N\widehat{\sigma}_x^2}{\sigma_x^2} = \sum_{i=1}^{N}\left(\frac{x_i - \widehat{m}_x}{\sigma_x}\right)^2$$

is a Chi-square random variable with $n-1$ degrees of freedom, which has a variance of $2(n-1)$. Therefore,

$$\mathrm{Var}\left(\frac{N\widehat{\sigma}_x^2}{\sigma_x^2}\right) = 2(N-1)$$

and, consequently, we have

$$\mathrm{Var}\left(\widehat{\sigma_x^2}\right) = 2\frac{\sigma_x^4}{N^2}(N-1)$$

3.2 Let $x(n)$ be a stationary random process with zero mean and autocorrelation $r_x(k)$. We form the process, $y(n)$, as follows

$$y(n) = x(n) + f(n)$$

where $f(n)$ is a known deterministic sequence. Find the mean $m_y(n)$ and the autocorrelation $r_y(k,l)$ of the process $y(n)$.

Solution

The mean of the process is

$$m_y(n) = E\{y(n)\} = E\{x(n)\} + f(n) = f(n)$$

and the autocorrelation is

$$\begin{aligned} r_y(k,l) &= E\{y(k)y(l)\} = E\{[x(k)+f(k)][x(l)+f(l)]\} \\ &= E\{x(k)x(l)\} + f(k)f(l) = r_x(k,l) + f(k)f(l) \end{aligned}$$

3.3 A discrete-time random process $x(n)$ is generated as follows,

$$x(n) = \sum_{k=1}^{p} a(k)x(n-k) + w(n)$$

where $w(n)$ is a white noise process with variance σ_w^2. Another process, $z(n)$, is formed by adding noise to $x(n)$,

$$z(n) = x(n) + v(n)$$

where $v(n)$ is white noise with a variance of σ_v^2 that is uncorrelated with $w(n)$.

(a) Find the power spectrum of $x(n)$.

(b) Find the power spectrum of $z(n)$.

Solution

(a) Since $x(n)$ is the output of an all-pole filter driven by white noise, $x(n)$ is an AR(p) process with a power spectrum

$$P_x(e^{j\omega}) = \frac{\sigma_w^2}{|A(e^{j\omega})|^2}$$

where

$$A(e^{j\omega}) = 1 - \sum_{k=1}^{p} a(k)e^{-jk\omega}$$

(b) The process $z(n)$ is a sum of two random processes

$$z(n) = x(n) + v(n)$$

Since $x(n)$ is a linear combination of values of $w(n)$,

$$x(n) = \sum_{k=-\infty}^{n} h(k)w(n-k)$$

where $h(n)$ is the unit sample response of the filter generating $x(n)$, and since $v(n)$ is uncorrelated with $w(n)$, then $v(n)$ is uncorrelated with $x(n)$, and we have

$$r_z(k) = r_x(k) + r_v(k)$$

Therefore,

$$P_z(e^{j\omega}) = P_x(e^{j\omega}) + P_v(e^{j\omega})$$

and

$$P_z(e^{j\omega}) = \frac{\sigma_w^2}{|A(e^{j\omega})|^2} + \sigma_v^2 = \frac{\sigma_w^2 + \sigma_v^2|A(e^{j\omega})|^2}{|A(e^{j\omega})|^2}$$

3.4 Suppose we are given a linear shift-invariant system having a system function

$$H(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

that is excited by zero mean exponentially correlated noise $x(n)$ with an autocorrelation sequence

$$r_x(k) = \left(\tfrac{1}{2}\right)^{|k|}$$

Let $y(n)$ be the output process, $y(n) = x(n) * h(n)$.

(a) Find the power spectrum, $P_y(z)$, of $y(n)$.

(b) Find the autocorrelation sequence, $r_y(k)$, of $y(n)$.

(c) Find the cross-correlation, $r_{xy}(k)$, between $x(n)$ and $y(n)$.

(d) Find the *cross-power spectral density*, $P_{xy}(z)$, which is the z transform of the cross-correlation $r_{xy}(k)$.

Solution

(a) The power spectrum of $x(n)$ is

$$P_x(z) = \frac{3/4}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z)}$$

and the power spectrum of $y(n)$ is

$$P_y(z) = H(z)H(z^{-1})P_x(z) = \frac{3/4}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{3}z)}$$

(b) The autocorrelation sequence for $y(n)$ may be easily found using the z-transform pair

$$\alpha^{|k|} \longleftrightarrow \frac{1 - \alpha^2}{(1 - \alpha z^{-1})(1 - \alpha z)}$$

Since

$$(\tfrac{1}{3})^{|k|} \longleftrightarrow \frac{8/9}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{3}z)}$$

then

$$r_y(k) = \tfrac{27}{32}(\tfrac{1}{3})^{|k|}$$

(c) The cross-correlation $r_{xy}(k)$ between $x(n)$ and $y(n)$ is

$$r_{xy}(k) = r_x(k) * h(-k)$$

This may be easily computed using z-transforms as follows,

$$\begin{aligned} P_{xy}(z) &= P_x(z)H(z^{-1}) = \frac{3/4}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z)} \cdot \frac{1 - \frac{1}{2}z}{1 - \frac{1}{3}z} \\ &= \frac{3/4}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z)} \end{aligned}$$

Writing this in terms of z^{-1} and performing a partial fraction expansion gives

$$P_{xy}(z) = \tfrac{3}{4}\frac{z^{-1}}{(1-\frac{1}{2}z^{-1})(z^{-1}-\frac{1}{3})} = \frac{9/10}{1-\frac{1}{2}z^{-1}} + \frac{3/10}{z^{-1}-\frac{1}{3}}$$

Inverse z-transforming gives

$$r_{xy}(k) = \tfrac{9}{10}(\tfrac{1}{2})^k u(k) + \tfrac{9}{10}(3)^{-k}u(-k-1)$$

(d) The cross-power spectral density, $P_{xy}(z)$, as computed in part (a), is

$$P_{xy}(z) = \frac{3/4}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z)}$$

(e) The cross-correlation, $r_{xy}(k)$, between $x(n)$ and $y(n)$ may found by computing the inverse z-transform of the cross-power spectral density,

$$P_{xy}(z) = \tfrac{3}{4}\frac{z^{-1}}{(1-\frac{1}{2}z^{-1})(z^{-1}-\frac{1}{3})} = \frac{\frac{9}{10}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{3}{10}}{z^{-1}-\frac{1}{3}}$$

Inverse transforming gives
