Solutions to B Problems

CHAPTER 2

<u>B-2-1.</u>

$$f(t) = 0 \qquad t < 0$$
$$= t\,e^{-2t} \qquad t \geq 0$$

Note that

$$\mathcal{L}[t] = \frac{1}{s^2}$$

Referring to Equation (2-2), we obtain

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[t\,e^{-2t}] = \frac{1}{(s+2)^2}$$

<u>B-2-2.</u>

(a)

$$f_1(t) = 0 \qquad t < 0$$
$$= 3\sin(5t + 45^\circ) \qquad t \geq 0$$

Note that

$$3\sin(5t + 45^\circ) = 3\sin 5t \cos 45^\circ + 3\cos 5t \sin 45^\circ$$
$$= \frac{3}{\sqrt{2}}\sin 5t + \frac{3}{\sqrt{2}}\cos 5t$$

So we have

$$F_1(s) = \mathcal{L}[f_1(t)] = \frac{3}{\sqrt{2}}\,\frac{5}{s^2 + 5^2} + \frac{3}{\sqrt{2}}\,\frac{s}{s^2 + 5^2}$$
$$= \frac{3}{\sqrt{2}}\,\frac{s + 5}{s^2 + 25}$$

(b)

$$f_2(t) = 0 \qquad t < 0$$
$$= 0.03(1 - \cos 2t) \qquad t \geq 0$$

$$F_2(s) = \mathcal{L}[f_2(t)] = 0.03\,\frac{1}{s} - 0.03\,\frac{s}{s^2 + 2^2} = \frac{0.12}{s(s^2 + 4)}$$

B-2-3.

$$f(t) = 0 \qquad t < 0$$
$$= t^2 e^{-at} \qquad t \geq 0$$

Note that

$$\mathcal{L}[t^2] = \frac{2}{s^3}$$

Referring to Equation (2-2), we obtain

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[t^2 e^{-at}] = \frac{2}{(s + a)^3}$$

B-2-4.

$$f(t) = 0 \qquad t < 0$$
$$= \cos 2\omega t \cos 3\omega t \qquad t \geq 0$$

Noting that

$$\cos 2\omega t \cos 3\omega t = \tfrac{1}{2}(\cos 5\omega t + \cos \omega t)$$

we have

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[\tfrac{1}{2}(\cos 5\omega t + \cos \omega t)]$$
$$= \frac{1}{2}\left(\frac{s}{s^2 + 25\omega^2} + \frac{s}{s^2 + \omega^2}\right) = \frac{(s^2 + 13\omega^2)s}{(s^2 + 25\omega^2)(s^2 + \omega^2)}$$

B-2-5. The function f(t) can be written as

$$f(t) = (t - a)\, 1(t - a)$$

The Laplace transform of f(t) is

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[(t - a)\, 1(t - a)] = \frac{e^{-as}}{s^2}$$

B-2-6.

$$f(t) = c\, 1(t - a) - c\, 1(t - b)$$

The Laplace transform of f(t) is

$$F(s) = c\,\frac{e^{-as}}{s} - c\,\frac{e^{-bs}}{s} = \frac{c}{s}\,(e^{-as} - e^{-bs})$$

B-2-7. The function f(t) can be written as

$$f(t) = \frac{10}{a^2} - \frac{12.5}{a^2} 1(t - \frac{a}{5}) + \frac{2.5}{a^2} 1(t - a)$$

So the Laplace transform of f(t) becomes

$$F(s) = \mathcal{L}[f(t)] = \frac{10}{a^2}\frac{1}{s} - \frac{12.5}{a^2}\frac{1}{s} e^{-(a/5)s} + \frac{2.5}{a^2}\frac{1}{s} e^{-as}$$

$$= \frac{1}{a^2 s}(10 - 12.5\, e^{-(a/5)s} + 2.5\, e^{-as})$$

As a approaches zero, the limiting value of F(s) becomes as follows:

$$\lim_{a\to 0} F(s) = \lim_{a\to 0} \frac{10 - 12.5\, e^{-(a/5)s} + 2.5\, e^{-as}}{a^2 s}$$

$$= \lim_{a\to 0} \frac{\frac{d}{da}(10 - 12.5\, e^{-(a/5)s} + 2.5\, e^{-as})}{\frac{d}{da} a^2 s}$$

$$= \lim_{a\to 0} \frac{2.5\, s\, e^{-(a/5)s} - 2.5\, s\, e^{-as}}{2as}$$

$$= \lim_{a\to 0} \frac{\frac{d}{da}(2.5\, e^{-(a/5)s} - 2.5\, e^{-as})}{\frac{d}{da} 2a}$$

$$= \lim_{a\to 0} \frac{-0.5\, s\, e^{-(a/5)s} + 2.5\, s\, e^{-as}}{2}$$

$$= \frac{-0.5\, s + 2.5\, s}{2} = \frac{2s}{2} = s$$

B-2-8. The function f(t) can be written as

$$f(t) = \frac{24}{a^3} t - \frac{24}{a^2} 1(t - \frac{a}{2}) - \frac{24}{a^3}(t - a)\, 1(t - a)$$

So the Laplace transform of f(t) becomes

$$F(s) = \frac{24}{a^3}\frac{1}{s^2} - \frac{24}{a^2}\frac{1}{s} e^{-\frac{1}{2}as} - \frac{24}{a^3}\frac{e^{-as}}{s^2}$$

$$= \frac{24}{a^3}\left(\frac{1}{s^2} - \frac{a}{s} e^{-\frac{1}{2}as} - \frac{e^{-as}}{s^2}\right)$$

The limiting value of F(s) as a approaches zero is

$$\lim_{a\to 0} F(s) = \lim_{a\to 0} \frac{24(1 - as\, e^{-\frac{1}{2}as} - e^{-as})}{a^3 s^2}$$

$$= \lim_{a\to 0} \frac{\frac{d}{da} 24(1 - as\, e^{-\frac{1}{2}as} - e^{-as})}{\frac{d}{da} a^3 s^2}$$

$$= \lim_{a\to 0} \frac{24(-s\, e^{-\frac{1}{2}as} + \frac{as^2}{2} e^{-\frac{1}{2}as} + s\, e^{-as})}{3a^2 s^2}$$

$$= \lim_{a\to 0} \frac{\frac{d}{da} 8(-\, e^{-\frac{1}{2}as} + \frac{as}{2} e^{-\frac{1}{2}as} + e^{-as})}{\frac{d}{da} a^2 s}$$

$$= \lim_{a\to 0} \frac{8\left[\frac{s}{2} e^{-\frac{1}{2}as} + \frac{s}{2} e^{-\frac{1}{2}as} + \frac{as}{2}\left(\frac{-s}{2}\right) e^{-\frac{1}{2}as} - se^{-as}\right]}{2as}$$

$$= \lim_{a\to 0} \frac{\frac{d}{da}(4\, e^{-\frac{1}{2}as} - as\, e^{-\frac{1}{2}as} - 4\, e^{-as})}{\frac{d}{da} a}$$

$$= \lim_{a\to 0} \frac{-2s\, e^{-\frac{1}{2}as} - s\, e^{-\frac{1}{2}as} + as\,\frac{s}{2} e^{-\frac{1}{2}as} + 4s\, e^{-as}}{1}$$

$$= -2s - s + 4s = s$$

B-2-9.

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$= \lim_{s\to 0} \frac{s\, 5(s + 2)}{s(s + 1)} = \frac{5 \times 2}{1} = 10$$

B-2-10.

$$f(0+) = \lim_{t\to 0+} f(t) = \lim_{s\to\infty} \frac{s\, 2(s + 2)}{s(s + 1)(s + 3)} = 0$$

B-2-11. Define

$$y = \dot{x}$$

Then

$$y(0+) = \dot{x}(0+)$$

The initial value of y can be obtained by use of the initial value theorem as follows:

$$y(0+) = \lim_{s \to \infty} sY(s)$$

Since

$$Y(s) = \mathcal{L}_+[y(t)] = \mathcal{L}_+[\dot{x}(t)] = sX(s) - x(0+)$$

we obtain

$$y(0+) = \lim_{s \to \infty} sY(s) = \lim_{s \to \infty} s[sX(s) - x(0+)]$$

$$= \lim_{s \to \infty} [s^2X(s) - sx(0+)]$$

B-2-12. Note that

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^2}{dt^2} f(t)\right] = s^2F(s) - sf(0) - \dot{f}(0)$$

Define

$$g(t) = \frac{d^2}{dt^2} f(t)$$

Then

$$\mathcal{L}\left[\frac{d^3}{dt^3} f(t)\right] = \mathcal{L}\left[\frac{d}{dt} g(t)\right] = sG(s) - g(0)$$

$$= s[s^2F(s) - sf(0) - \dot{f}(0)] - \ddot{f}(0)$$

$$= s^3F(s) - s^2f(0) - s\dot{f}(0) - \ddot{f}(0)$$

B-2-13.

$$\int_0^T f(t)\, e^{-st}\, dt = \int_0^b a\, e^{-st}\, dt = a \left. \frac{e^{-st}}{-s} \right|_0^b$$

$$= a\left(\frac{e^{-bs} - 1}{-s}\right) = a\left(\frac{1 - e^{-bs}}{s}\right)$$

Referring to Problem A-2-12, we have

$$F(s) = \frac{\int_0^T f(t)\, e^{-st}\, dt}{1 - e^{-Ts}} = \frac{a(1 - e^{-bs})}{s(1 - e^{-Ts})}$$

<u>B-2-14</u>.

$$\int_{0-}^{T-} f(t)\, e^{-st}\, dt = \int_{0-}^{0+} \delta(t)\, e^{-st}\, dt = 1$$

Referring to Problem A-2-12, we obtain

$$F(s) = \frac{1}{1 - e^{-Ts}}$$

<u>B-2-15</u>.

(a)
$$F_1(s) = \frac{s + 5}{(s + 1)(s + 3)} = \frac{a_1}{s + 1} + \frac{a_2}{s + 3}$$

where

$$a_1 = \left.\frac{s + 5}{s + 3}\right|_{s = -1} = \frac{4}{2} = 2$$

$$a_2 = \left.\frac{s + 5}{s + 1}\right|_{s = -3} = \frac{2}{-2} = -1$$

$F_1(s)$ can thus be written as

$$F_1(s) = \frac{2}{s + 1} - \frac{1}{s + 3}$$

and the inverse Laplace transform of $F_1(s)$ is

$$f_1(t) = 2\, e^{-t} - e^{-3t}$$

(b)
$$F_2(s) = \frac{3(s + 4)}{s(s + 1)(s + 2)} = \frac{a_1}{s} + \frac{a_2}{s + 1} + \frac{a_3}{s + 2}$$

where

$$a_1 = \left.\frac{3(s + 4)}{(s + 1)(s + 2)}\right|_{s = 0} = \frac{3 \times 4}{2} = 6$$

$$a_2 = \left.\frac{3(s + 4)}{s(s + 2)}\right|_{s = -1} = \frac{3 \times 3}{(-1) \times 1} = -9$$

$$a_3 = \frac{3(s + 4)}{s(s + 1)}\bigg|_{s = -2} = \frac{3 \times 2}{(-2)(-1)} = 3$$

$F_2(s)$ can thus be written as

$$F_2(s) = \frac{6}{s} - \frac{9}{s + 1} + \frac{3}{s + 2}$$

and the inverse Laplace transform of $F_2(s)$ is

$$f_2(t) = 6 - 9\, e^{-t} + 3\, e^{-2t}$$

B-2-16.

(a)
$$F_1(s) = \frac{6s + 3}{s^2} = \frac{6}{s} + \frac{3}{s^2}$$

The inverse Laplace transform of $F_1(s)$ is

$$f_1(t) = 6 + 3t$$

(b)
$$F_2(s) = \frac{5s + 2}{(s + 1)(s + 2)^2} = \frac{a}{s + 1} + \frac{b_2}{(s + 2)^2} + \frac{b_1}{s + 2}$$

where

$$a = \frac{5s + 2}{(s + 2)^2}\bigg|_{s = -1} = \frac{-5 + 2}{1^2} = -3$$

$$b_2 = \frac{5s + 2}{s + 1}\bigg|_{s = -2} = \frac{-10 + 2}{-2 + 1} = 8$$

$$b_1 = \frac{d}{ds}\left(\frac{5s + 2}{s + 1}\right)\bigg|_{s = -2} = \frac{5(s + 1) - (5s + 2)}{(s + 1)^2}\bigg|_{s = -2}$$

$$= \frac{5(-1) - (-10 + 2)}{1^2} = 3$$

$F_2(s)$ can thus be written as

$$F_2(s) = \frac{-3}{s + 1} + \frac{8}{(s + 2)^2} + \frac{3}{s + 2}$$

and the inverse Laplace transform of $F_2(s)$ is

$$f_2(t) = -3\ e^{-t} + 8t\ e^{-2t} + 3\ e^{-2t}$$

B-2-17.

$$F(s) = \frac{2s^2 + 4s + 5}{s(s + 1)} = 2 + \frac{2}{s + 1} + \frac{5}{s(s + 1)}$$

$$= 2 + \frac{2}{s + 1} + \frac{5}{s} - \frac{5}{s + 1} = 2 - \frac{3}{s + 1} + \frac{5}{s}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = 2\ \delta(t) - 3\ e^{-t} + 5$$

B-2-18.

$$F(s) = \frac{s^2 + 2s + 4}{s^2} = 1 + \frac{2}{s} + \frac{4}{s^2}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \delta(t) + 2 + 4t$$

B-2-19.

$$F(s) = \frac{s}{s^2 + 2s + 10} = \frac{s + 1 - 1}{(s + 1)^2 + 3^2}$$

$$= \frac{s + 1}{(s + 1)^2 + 3^2} - \frac{3}{(s + 1)^2 + 3^2}\ \frac{1}{3}$$

Hence

$$f(t) = e^{-t} \cos 3t - \frac{1}{3}\ e^{-t} \sin 3t$$

B-2-20.

$$F(s) = \frac{s^2 + 2s + 5}{s^2(s + 1)} = \frac{a}{s^2} + \frac{b}{s} + \frac{c}{s + 1}$$

where

$$a = \left.\frac{s^2 + 2s + 5}{s + 1}\right|_{s = 0} = 5$$

$$b = \frac{(2s + 2)(s + 1) - (s^2 + 2s + 5)}{(s + 1)^2}\bigg|_{s = 0} = \frac{2 - 5}{1} = -3$$

$$c = \frac{s^2 + 2s + 5}{s^2}\bigg|_{s = -1} = \frac{1 - 2 + 5}{1} = 4$$

Hence

$$F(s) = \frac{5}{s^2} + \frac{-3}{s} + \frac{4}{s + 1}$$

The inverse Laplace transform of F(s) is

$$f(t) = 5t - 3 + 4\, e^{-t}$$

B-2-21.

$$F(s) = \frac{2s + 10}{(s + 1)^2(s + 4)} = \frac{a}{(s + 1)^2} + \frac{b}{s + 1} + \frac{c}{s + 4}$$

where

$$a = \frac{2s + 10}{s + 4}\bigg|_{s = -1} = \frac{-2 + 10}{3} = \frac{8}{3}$$

$$b = \frac{2(s + 4) - (2s + 10)}{(s + 4)^2}\bigg|_{s = -1} = \frac{6 - 8}{3^2} = \frac{-2}{9}$$

$$c = \frac{2s + 10}{(s + 1)^2}\bigg|_{s = -4} = \frac{-8 + 10}{9} = \frac{2}{9}$$

Hence

$$F(s) = \frac{8}{3} \cdot \frac{1}{(s + 1)^2} - \frac{2}{9}\, \frac{1}{s + 1} + \frac{2}{9}\, \frac{1}{s + 4}$$

The inverse Laplace transform of F(s) is

$$f(t) = \frac{8}{3}\, te^{-t} - \frac{2}{9}\, e^{-t} + \frac{2}{9}\, e^{-4t}$$

B-2-22.

$$F(s) = \frac{1}{s^2(s^2 + \omega^2)} = \left(\frac{1}{s^2} - \frac{1}{s^2 + \omega^2}\right)\frac{1}{\omega^2}$$

The inverse Laplace transform of F(s) is

$$f(t) = \frac{1}{\omega^2}\left(t - \frac{1}{\omega}\sin \omega t\right)$$

B-2-23.

$$F(s) = \frac{c}{s^2}(1 - e^{-as}) - \frac{b}{s}e^{-as} \qquad a > 0$$

The inverse Laplace transform of F(s) is

$$f(t) = ct - c(t - a)1(t - a) - b\,1(t - a)$$

B-2-24. A MATLAB program to obtain partial-fraction expansions of the given function F(s) is given below.

```
num = [0  0  0  0  1];
den = [1  3  2  0  0];
[r,p,k] = residue(num,den)

r =

  -0.2500
   1.0000
  -0.7500
   0.5000


p =

  -2
  -1
   0
   0


k =

   []
```

From this computer output we obtain

$$F(s) = \frac{1}{s^4 + 3s^3 + 2s^2} = \frac{-0.25}{s + 2} + \frac{1}{s + 1} + \frac{-0.75}{s} + \frac{0.5}{s^2}$$

The inverse Laplace transform of F(s) is

$$f(t) = -0.25\, e^{-2t} + e^{-t} - 0.75 + 0.5t$$

B-2-25. A possible MATLAB program to obtain partial-fraction expansions of the given function F(s) is given below.

```
num = [0  0  3  4  1];
den = [1  2  5  8  10];
[r,p,k] = residue(num,den)

r =

  0.3661 - 0.4881i
  0.3661 + 0.4881i
 -0.3661 - 0.0006i
 -0.3661 + 0.0006i


p =

  0.2758 + 1.9081i
  0.2758 - 1.9081i
 -1.2758 + 1.0309i
 -1.2758 - 1.0309i


k =

   []
```

From this computer output we obtain

$$F(s) = \frac{3s^2 + 4s + 1}{s^4 + 2s^3 + 5s^2 + 8s + 10}$$

$$= \frac{0.3661 - j0.4881}{s - 0.2758 - j1.9081} + \frac{0.3661 + j0.4881}{s - 0.2758 + j1.9081}$$

$$+ \frac{-0.3661 - j0.0006}{s + 1.2758 - j1.0309} + \frac{-0.3661 + j0.0006}{s + 1.2758 + j1.0309}$$

Since the poles are complex quantities, we may rewrite F(s) as follows:

$$F(s) = \frac{0.7322s + 1.6607}{(s - 0.2758)^2 + 1.9081^2} + \frac{-0.7322s - 0.9329}{(s + 1.2758)^2 + 1.0309^2}$$

$$= \frac{0.7322(s - 0.2758) + 1.9081 \times 0.9762}{(s - 0.2758)^2 + 1.9081^2}$$

$$+ \frac{-0.7322(s + 1.2758) + 1.0309 \times 0.001204}{(s + 1.2758)^2 + 1.0309^2}$$

Then, the inverse Laplace transform of F(s) is obtained as

$$\begin{aligned} f(t) = {} & 0.7322\ e^{0.2758t} \cos 1.9081t \\ & + 0.9762\ e^{0.2758t} \sin 1.9081t \\ & - 0.7322\ e^{-1.2758t} \cos 1.0309t \\ & + 0.001204\ e^{-1.2758t} \sin 1.0309t \end{aligned}$$

B-2-26. $\ddot{x} + 4x = 0, \quad x(0) = 5, \quad \dot{x}(0) = 0$

The Laplace transform of the given differential equation is

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 4X(s) = 0$$

Substitution of the initial conditions into this last equation gives

$$(s^2 + 4)X(s) = 5s$$

Solving for X(s), we obtain

$$X(s) = \frac{5s}{s^2 + 4}$$

The inverse Laplace transform of X(s) is

$$x(t) = 5 \cos 2t$$

This is the solution of the given differential equation.

B-2-27. $\ddot{x} + \omega_n^2 x = t, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$

The Laplace transform of this differential equation is

$$s^2X(s) + \omega_n^2 X(s) = \frac{1}{s^2}$$

Solving this equation for X(s), we obtain

$$X(s) = \frac{1}{s^2(s^2 + \omega_n^2)} = \left(\frac{1}{s^2} - \frac{1}{s^2 + \omega_n^2}\right)\frac{1}{\omega_n^2}$$

The inverse Laplace transform of X(s) is

$$x(t) = \frac{1}{\omega_n^2}\left(t - \frac{1}{\omega_n}\sin\omega_n t\right)$$

This is the solution of the given differential equation.

B-2-28. $2\ddot{x} + 2\dot{x} + x = 1, \qquad x(0) = 0, \qquad \dot{x}(0) = 2$

The Laplace transform of this differential equation is

$$2[s^2X(s) - sx(0) - \dot{x}(0)] + 2[sX(s) - x(0)] + X(s) = \frac{1}{s}$$

Substitution of the initial conditions into this equation gives

$$2[s^2X(s) - 2] + 2[sX(s)] + X(s) = \frac{1}{s}$$

or

$$(2s^2 + 2s + 1)X(s) = 4 + \frac{1}{s}$$

Solving this last equation for X(s), we get

$$X(s) = \frac{4s + 1}{s(2s^2 + 2s + 1)}$$

$$= \frac{4}{2s^2 + 2s + 1} + \frac{1}{s(2s^2 + 2s + 1)}$$

$$= \frac{2}{(s + 0.5)^2 + 0.25} + \frac{0.5}{s[(s + 0.5)^2 + 0.25]}$$

$$= \frac{4 \times 0.5}{(s + 0.5)^2 + 0.5^2} + \frac{1}{s} - \frac{(s + 0.5) + 0.5}{(s + 0.5)^2 + 0.5^2}$$

The inverse Laplace transform of X(s) gives

$$x(t) = 4\, e^{-0.5t} \sin 0.5t + 1 - e^{-0.5t} \cos 0.5t - e^{-0.5t} \sin 0.5t$$

$$= 1 + 3\, e^{-0.5t} \sin 0.5t - e^{-0.5t} \cos 0.5t$$

B-2-29. $\qquad 2\ddot{x} + 7\dot{x} + 3x = 0, \qquad x(0) = 3, \qquad \dot{x}(0) = 0$

Taking the Laplace transform of this differential equation, we obtain

$$2[s^2X(s) - sx(0) - \dot{x}(0)] + 7[sX(s) - x(0)] + 3X(s) = 0$$

By substituting the given initial conditions into this last equation,

$$2[s^2X(s) - 3s] + 7[sX(s) - 3] + 3X(s) = 0$$

or

$$(2s^2 + 7s + 3)X(s) = 6s + 21$$

Solving for X(s) yields

$$X(s) = \frac{6s + 21}{2s^2 + 7s + 3} = \frac{6s + 21}{(2s + 1)(s + 3)}$$

$$= \frac{7.2}{2s + 1} - \frac{0.6}{s + 3} = \frac{3.6}{s + 0.5} - \frac{0.6}{s + 3}$$

Finally, taking the inverse Laplace transform of X(s), we obtain

$$x(t) = 3.6\, e^{-0.5t} - 0.6\, e^{-3t}$$

B-2-30. $\qquad \ddot{x} + x = \sin 3t, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$

The Laplace transform of this differential equation is

$$s^2X(s) + X(s) = \frac{3}{s^2 + 3^2}$$

Solving this equation for X(s), we get

$$X(s) = \frac{3}{(s^2 + 1)(s^2 + 9)} = \frac{3}{8}\frac{1}{s^2 + 1} - \frac{1}{8}\frac{3}{s^2 + 9}$$

The inverse Laplace transform of X(s) gives

$$x(t) = \frac{3}{8} \sin t - \frac{1}{8} \sin 3t$$

CHAPTER 3

B-3-1. $J = \frac{1}{2}\, mR^2 = \frac{1}{2} \times 100 \times 0.5^2 = 12.5 \text{ kg-m}^2$

B-3-2. Assume that the body of known moment of inertia J_0 is turned through a small angle θ about the vertical axis and then released. The equation of motion for the oscillation is

$$J_0 \ddot{\theta} = -k\theta$$

where k is the torsional spring constant of the string. This equation can be written as

$$\ddot{\theta} + \frac{k}{J_0}\theta = 0$$

or

$$\ddot{\theta} + \omega_n^2 \theta = 0$$

where

$$\omega_n = \sqrt{\frac{k}{J_0}}$$

The period T_0 of this oscillation is

$$T_0 = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{k}{J_0}}} \tag{1}$$

Next, we attach a rotating body of unknown moment of inertia J and measure the period T of oscillation. The equation for the period T is

$$T = \frac{2\pi}{\sqrt{\frac{k}{J}}} \tag{2}$$

By eliminating the unknown torsional spring constant k from Equations (1) and (2), we obtain

$$\frac{2\pi\sqrt{J_0}}{T_0} = \frac{2\pi\sqrt{J}}{T}$$

Hence

$$J = J_0\left(\frac{T}{T_0}\right)^2 \tag{3}$$

The unknown moment of inertia J can therefore be determined by measuring the period of oscillation T and substituting it into Equation (3).

B-3-3. Define the vertical displacement of the ball as $x(t)$ with $x(0) = 0$. The positive direction is downward. The equation of motion for the system is

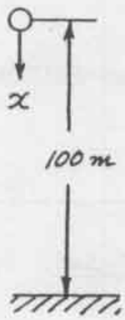
$$m\ddot{x} = mg$$

with initial conditions $x(0) = 0$ m and $\dot{x}(0) = 20$ m/s. So we have

$$\ddot{x} = g$$

$$\dot{x} = gt + \dot{x}(0)$$

$$x = \tfrac{1}{2} gt^2 + \dot{x}(0)t + x(0) = \tfrac{1}{2} gt^2 + 20\, t$$



Assume that at $t = t_1$ the ball reaches the ground. Then

$$100 = \tfrac{1}{2} \times 9.81\, t_1{}^2 + 20\, t_1$$

from which we obtain

$$t_1 = 2.915 \text{ s}$$

The ball reaches the ground in 2.915 s.

B-3-4. Define the torque applied to the flywheel as T. The equation of motion for the system is

$$J\ddot{\theta} = T, \qquad \theta(0) = 0, \qquad \dot{\theta}(0) = 0$$

from which we obtain

$$\dot{\theta} = \frac{T}{J}\, t$$

By substituting numerical values into this equation, we have

$$20 \times 6.28 = \frac{T}{50} \times 5$$

Thus

$$T = 1256 \text{ N-m}$$

B-3-5. $$J\ddot{\theta} = -T \qquad (T = \text{braking torque})$$

Integrating this equation,

$$\dot{\theta} = -\frac{T}{J}\, t + \dot{\theta}(0), \qquad \dot{\theta}(0) = 100 \text{ rad/s}$$

Substituting the given numerical values,

$$20 = -\frac{T}{J} \times 15 + 100$$

Solving for T/J, we obtain

$$\frac{T}{J} = 5.33$$

Hence, the deceleration given by the brake is 5.33 rad/s^2.

The total angle rotated in 15-second period is obtained from

$$\theta(t) = -\frac{T}{J}\frac{t^2}{2} + \dot{\theta}(0)t + \theta(0), \qquad \theta(0) = 0, \qquad \dot{\theta}(0) = 100$$

as follows;

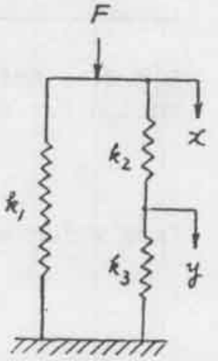
$$\theta(15) = -5.33 \times \frac{15^2}{2} + 100 \times 15 = 900 \text{ rad}$$

B-3-6. Assume that we apply force F to the spring system. Then

$$F = k_1 x + k_2(x - y)$$

$$k_2(x - y) = k_3 y$$

Eliminating y from the preceding equations, we obtain



$$F = \frac{k_1(k_2 + k_3) + k_2 k_3}{k_2 + k_3}\, x$$

$$= \left(k_1 + \frac{1}{\frac{1}{k_2} + \frac{1}{k_3}} \right) x$$

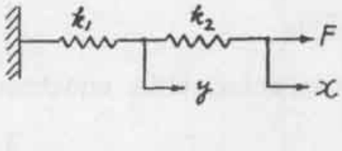
Hence the equivalent spring constant k_{eq} is given by

$$k_{eq} = k_1 + \frac{1}{\frac{1}{k_2} + \frac{1}{k_3}}$$

B-3-7. The equations for the system are

$$F = k_2(x - y)$$

$$k_2(x - y) = k_1 y$$



Eliminating y from the two equations gives

$$F = k_2 \left(x - \frac{k_2 x}{k_1 + k_2} \right) = \frac{k_1 k_2}{k_1 + k_2}\, x = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}\, x$$

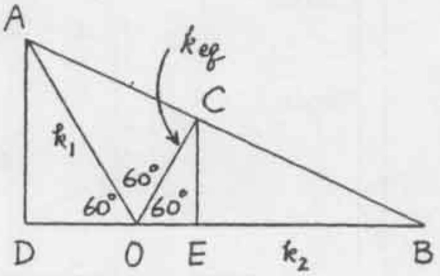
The equivalent spring constant k_{eq} is then obtained as

$$k_{eq} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

Next, consider the figure shown below. Note that ΔABD and ΔCBE are similar. So we have

$$\frac{\overline{CE}}{\overline{AD}} = \frac{\overline{BE}}{\overline{BD}}$$

which can be rewritten as



$$\frac{\overline{OC}\,\frac{\sqrt{3}}{2}}{\overline{OA}\,\frac{\sqrt{3}}{2}} = \frac{\overline{OB} - \overline{OC}\,\frac{1}{2}}{\overline{OB} + \overline{OA}\,\frac{1}{2}}$$

or

$$\overline{OC}(\overline{OB} + \tfrac{1}{2}\,\overline{OA}) = \overline{OA}(\overline{OB} - \tfrac{1}{2}\,\overline{OC})$$

Solving for $\overline{OC}$, we obtain

$$\overline{OC} = \frac{\overline{OA}\;\overline{OB}}{\overline{OA} + \overline{OB}} = \frac{1}{\frac{1}{\overline{OA}} + \frac{1}{\overline{OB}}} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} = k_{eq}$$

B-3-8.

(a) The force f due to the dampers is

$$f = b_1(\dot{y} - \dot{x}) + b_2(\dot{y} - \dot{x}) = (b_1 + b_2)(\dot{y} - \dot{x})$$

In terms of the equivalent viscous friction coefficient b_{eq}, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

Hence

$$b_{eq} = b_1 + b_2$$

(b) The force f due to the dampers is

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) \qquad (1)$$

where z is the displacement of a point between damper b_1 and damper b_2. (Note that the same force is transmitted through the shaft.) From Equation (1), we have

$$(b_1 + b_2)\dot{z} = b_2\dot{y} + b_1\dot{x}$$

or

$$\dot{z} = \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x}) \qquad (2)$$

In terms of the equivalent viscous friction coefficient b_{eq}, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

By substituting Equation (2) into Equation (1), we have

$$f = b_2(\dot{y} - \dot{z}) = b_2[\dot{y} - \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x})]$$

$$= \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x})$$

Thus,

$$f = b_{eq}(\dot{y} - \dot{x}) = b_2(\dot{y} - \dot{z}) = \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x})$$

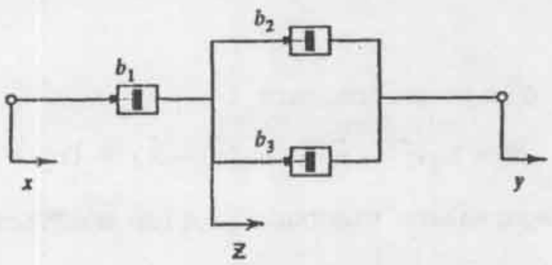
Hence,

$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}}$$

B-3-9. Since the same force transmits the shaft, we have

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) + b_3(\dot{y} - \dot{z}) \qquad (1)$$

where displacement z is defined in the figure below.



In terms of the equivalent viscous friction coefficient, the force f is given by

$$f = b_{eq}(\dot{y} - \dot{x}) \qquad (2)$$

From Equation (1) we have

$$b_1\dot{z} + b_2\dot{z} + b_3\dot{z} = b_1\dot{x} + b_2\dot{y} + b_3\dot{y}$$

or

$$\dot{z} = \frac{1}{b_1 + b_2 + b_3}[b_1\dot{x} + (b_2 + b_3)\dot{y}] \qquad (3)$$

By substituting Equation (3) into Equation (1), we have