SOLUTIONS TO THE CLASS 1 AND CLASS 2 PROBLEMS IN

TRANSPORT PHENOMENA

R. BYRON BIRD
WARREN E. STEWART
EDWIN N. LIGHTFOOT

Department of Chemical Engineering
University of Wisconsin
Madison, Wisconsin

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CHAPTER 1 — Checked by T. J. Sadowski

1.A Computation of Viscosities of Gases at Low Density

Two methods of solution are given in the text: the kinetic theory method based on Eq. 1.4-18, and the corresponding states method based on Fig. 1.3-1. The kinetic theory method is more accurate. Calculations by both methods are summarized here.

Kinetic Theory Method:

Gas	Constants from Table B-1			$\kappa T/\epsilon = \frac{293.2}{\epsilon/\kappa}$	Ω_μ from Table B-2	Predicted viscosity (cp) $\mu = 2.6693\times10^{-3}\frac{\sqrt{MT}}{\sigma^2\Omega_\mu}$
	M	σ (Å)	ε/κ (°K)			
O_2	32.00	3.433	113.	2.60	1.081	0.02029
N_2	28.02	3.681	91.5	3.20	1.022	0.01747
CH_4	16.04	3.822	137.	2.14	1.149	0.01091

Corresponding States Method:

Gas	Constants from Table B-1			$T_r = \frac{293.2}{T_c}$	$p_r = \frac{1.00}{p_c}$	μ/μ_c (Fig. 1.3-1)	Predicted Viscosity μ (cp)
	T_c (°K)	p_c (atm)	μ_c (cp)				
O_2	154.4	49.7	0.0250	1.90	0.020	0.82	0.0205
N_2	126.2	33.5	0.0180	2.32	0.030	0.97	0.0175
CH_4	191.0	45.8	0.0159	1.54	0.022	0.69	0.0110

The observed values are: O_2 (0.0203 cp), N_2 (0.0175 cp), CH_4 (0.0109 cp), as given in Table 1.1-2.

1.B Calculation of Viscosities of Gas Mixtures at Low Density

For the binary system being considered

i	Species	$\mu_i\times10^6$	M_i
1	H_2	88.4	2.016
2	CCl_2F_2	124.0	120.9

According to Eq. 1.4-20, $\Phi_{11}=1$, $\Phi_{22}=1$, and

$$\Phi_{12} = \frac{1}{\sqrt{8}}\left[1+\frac{2.016}{120.9}\right]^{-1/2}\left[1+\left(\frac{88.4}{124.0}\right)^{1/2}\left(\frac{120.9}{2.016}\right)^{1/4}\right]^2 = 3.934$$

$$\Phi_{21} = \frac{1}{\sqrt{8}}\left[1+\frac{120.9}{2016}\right]^{-1/2}\left[1+\left(\frac{124.0}{88.4}\right)^{1/2}\left(\frac{2.016}{120.9}\right)^{1/4}\right]^2 = 0.0920$$

Application of Eq. 1.4-19 then gives:

x_i	$\Sigma_1 = \Sigma x_j \Phi_{1j}$	$\Sigma_2 = \Sigma x_j \Phi_{2j}$	$A = x_1\mu_1/\Sigma_1$	$B = x_2\mu_2/\Sigma_2$	$A+B = \mu_{calc}\times10^6$	$\mu_{obs}\times10^6$
0.00	3.934	1.000	0.0	124.0	(124.0)	124.0
0.25	3.200	0.773	6.9	120.3	127.1	128.1
0.50	2.467	0.546	17.9	113.6	131.5	131.9
0.75	1.734	0.319	38.2	97.2	135.5	135.1
1.00	1.000	0.092	88.4	0.0	(88.4)	88.4

1.C Estimation of Viscosity of a Gas at High Density

(a.) From Table B-1 we find that $T_c = 126.2\,°K$, $p_c = 33.5$ atm., and $\mu_c = 180 \times 10^{-6}$ g cm⁻¹ sec⁻¹. Hence:

$$\left.\begin{aligned} p_r &= p/p_c = (1000+14.7)/(33.5)(14.7) = 1.40 \\ T_r &= T/T_c = (460+68)/(126.2)(1.8) = 2.32 \end{aligned}\right\}\text{From Fig. 1.3-1, } \mu_r = 1.07$$

Hence the predicted viscosity is $\mu = \mu_r\mu_c = (1.07)(180\times10^{-6}) = 1.93\times10^{-4}$ g cm⁻¹ sec⁻¹

(b.) From Table 1.1-2, at 20°C (68°F), $\mu^0 = 1.76\times10^{-4}$ g cm⁻¹ sec⁻¹. From Fig. 1.3-2 for the values of p_r and T_r calculated above, $\mu^\# = 1.1$. Hence the predicted viscosity is:

$$\mu = \mu^\#\mu^0 = (1.1)(1.75\times10^{-4}) = 1.93\times10^{-4} \text{ g cm}^{-1}\text{sec}^{-1}$$

1.D Estimation of Liquid Viscosity

(a) Equation 1.5-10 may be used with values of $\tilde{N}$, h, R from Table C.2, and

T (°K)	273.2	373.2
ρ (g cm⁻³)	0.9998	0.9584
$\tilde{V} = M/\rho$ (cm³ g-mole⁻¹)	18.02	18.80
$\Delta\hat{U}_{vap}$ (cal g⁻¹) at normal boiling pt.	498.9	498.9
$\Delta\tilde{U}_{vap} = M\Delta\hat{U}_{vap}$ (cal g-mole⁻¹)	8,988	8,988
$\exp\ 0.408\,\Delta\tilde{U}_{vap}/RT$	8.60×10^2	1.40×10^2
$\tilde{N}h/\tilde{V}$ (g cm⁻¹ sec⁻¹)	2.21×10^{-4}	2.12×10^{-4}
Predicted Viscosity (g cm⁻¹ sec⁻¹)	0.19	0.0297

(b) Use Eq. 1.5-12 to get: (Here $T_b = 373.2°K$)

At 273.2°K $\mu = (2.21 \times 10^{-4}) \exp (3.8)(373.2)/(273.2) = 4.0 \times 10^{-2} g\, cm^{-1} sec^{-1}$

At 373.2°K $\mu = (2.12 \times 10^{-4}) \exp (3.8) = 9.5 \times 10^{-3} g\, cm^{-1} sec^{-1}$

Summary of results:

	0°C	100°C
Observed viscosity (cp)	1.787	0.2821
Predicted by Eq. 1.5-10	48.4	2.97
Predicted by Eq. 1.5-12	4.0	0.95

Both equations give poor predictions. This is not surprising inasmuch as the empirical formula in Eqs. 1.5-9 *et seq.* do not hold for water, nor for most associated substances.

1.E Molecular Velocity and Mean Free Path

From Eq. 1.4-1 the mean speed of a molecule is

$$\bar{u} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{8\,(8.314 \times 10^{7})(273.2)}{\pi\,(32.00)}} = 4.25 \times 10^{4} \text{ cm sec}^{-1}$$

From Eq. 1.4-3 the mean free path is:

$$\lambda = \frac{RT}{\sqrt{2}\,\pi d^2 p \tilde{N}} = \frac{(82.057)(273.2)}{\sqrt{2}\,\pi\,(3 \times 10^{-8})^2 (1)(6.023 \times 10^{23})} = 9.3 \times 10^{-6} \text{ cm}$$

Hence the mean free path is $(9.3 \times 10^{-6})/(3 \times 10^{-8}) = 310$ molecular diameters under these conditions. In the liquid state, on the other hand, the corresponding ratio would be of the order of magnitude of, or even less than, unity.

1.F Comparison of the Uyehara-Watson Chart with Kinetic Theory

(a) Combination of Eqs. 1.4-11 and 18 gives.

$$\frac{\mu \sigma^2}{\sqrt{MT_c}} = 2.6693 \times 10^{-5} \frac{\sqrt{T_r}}{\Omega_\mu}$$

In addition, from Eq. 1.4-11 we know that $T_r = 0.77\, \kappa T/\epsilon$

The calculations for the kinetic theory plot are summarized below:

$\kappa T/\epsilon$	T_r	Ω_μ from Table B-2	Ordinate = $2.6693\sqrt{T_r}/\Omega_\mu$	μ_r from Fig. 1-3-1
0.3	0.231	2.785	0.461	—
0.4	0.308	2.492	0.595	—
0.5	0.385	2.257	0.735	—
0.8	0.616	1.780	1.178	0.275
1.0	0.770	1.587	1.492	0.342
1.5	1.155	1.314	2.185	0.515
2.0	1.54	1.175	2.82	0.69
3.0	2.31	1.039	3.90	0.96
5.0	3.85	0.927	5.65	1.38
8.0	6.16	0.854	7.76	1.88
10.0	7.70	0.824	9.00	2.16

As may be seen in the figure both the kinetic theory plot and the Uyehara-Watson plot predict essentially the same temperature dependence of viscosity, as is seen by the similar shapes of the two curves.

(b) Using an average value for the ratio of the ordinates of the two curves, we get:

$$\frac{\mu\sigma^2 M^{-1/2} T_c^{-1/2}}{\mu/\mu_c} \doteq 4.2 \times 10^{-5}$$

where μ [=] g cm^{-1} sec^{-1}, σ [=] Å, and T_c [=] °K. Solving for μ_c in micropoises we get:

$$\mu_c \doteq 42\ M^{1/2} T_c^{1/2} \sigma^{-2}$$

(c) Estimation of σ by Eq. 1.4-12 gives:

$$\mu_c \doteq 42\ M^{1/2}\ T_c^{1/2}\ \left[0.841\ \tilde{V}_c^{1/3}\right]^{-2} = 59.4\ M^{1/2} T_c^{1/2} \tilde{V}_c^{-2/3}$$

Eq. 1.3-2 gives a coefficient of 61.6 but is otherwise identical.

Estimation of σ by Eq. 1.4-13 gives:

$$\mu_c = 42\ M^{1/2}\ T_c^{1/2}\ \left[2.44\ (T_c/p_c)^{1/3}\right]^{-2}$$
$$= 7.06\ M^{1/2}\ p_c^{2/3}\ T_c^{-1/6}$$

This should be compared with Eq. 1.3-3 in which the numerical coefficient is 7.70. In view of the approximate nature of Eqs. 1.4-11, 12, 13, Uyehara and Watson's relations are to be preferred.

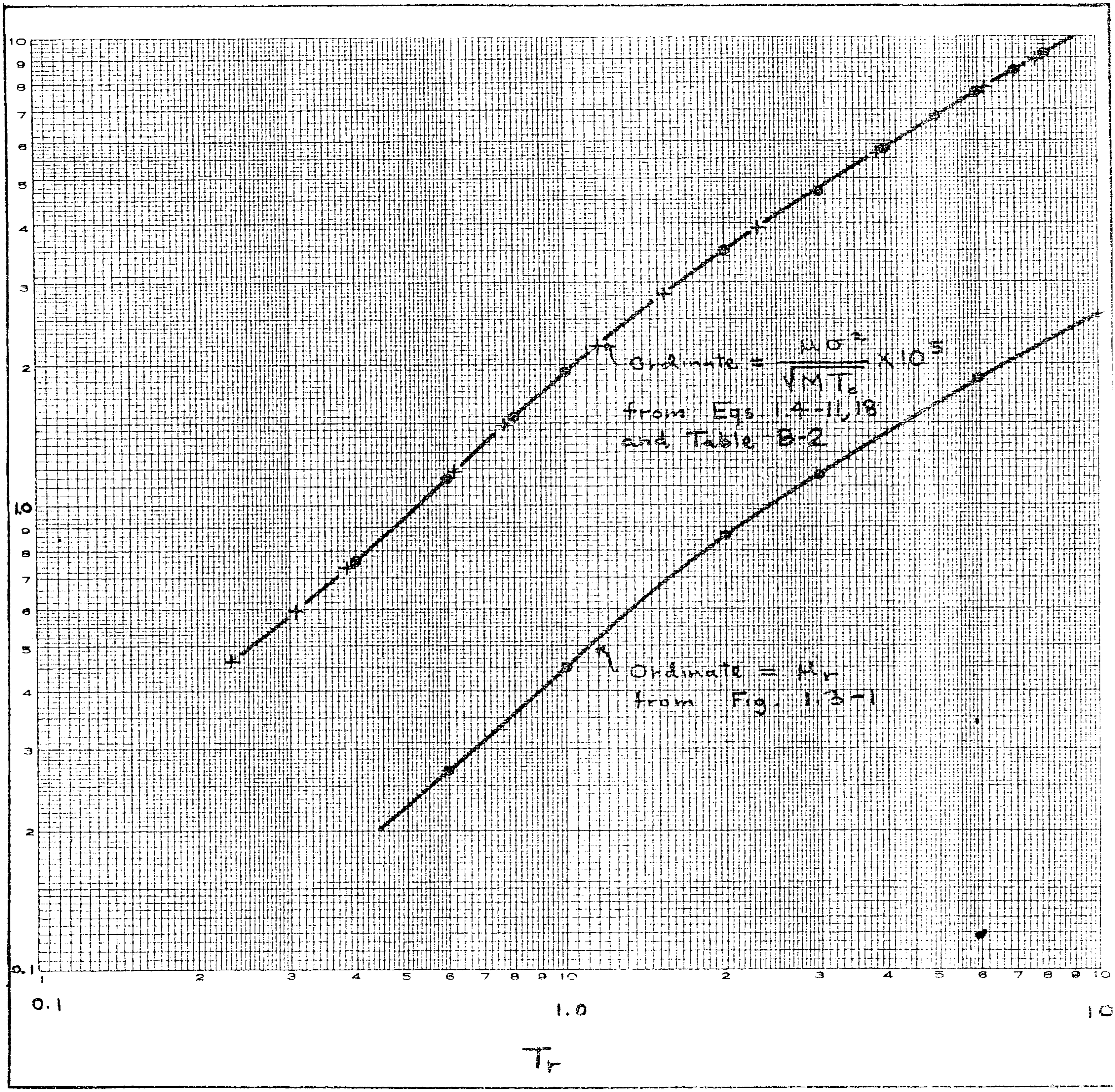

Figure for Problem 1.F

1.G Comparison of the Simple Kinetic Theory with the Exact Theory for Rigid Spheres

When Eq. 1.4-9 is written in terms of the units used in Eq. 1.4-18, and d is assumed to be the same as σ we get:

$$\mu = \frac{2}{3\pi^{3/2}} \frac{\sqrt{(M/\tilde{N})\kappa T}}{d^2}$$

$$= \frac{2}{3\pi^{3/2}} \left(\frac{1.3805 \times 10^{-16}}{6.023 \times 10^{23}}\right)^{1/2} \frac{\sqrt{MT}}{(10^{-8}\ d)^2}$$

$$= 1.813 \times 10^{-5} \frac{\sqrt{MT}}{\sigma^2}$$

Eq. 1.4-18 reduces to the same form as that above (by setting Ω_μ equal to unity) but the numerical constant is 2.6693×10^{-5}. Hence the simple kinetic theory is low by about 32% for rigid spheres.

CHAPTER 2 – Checked by V. D Shah

2.A Determination of Capillary Radius by Flow Measurements

Solve the Hagen-Poiseuille formula for R to get

$$R = \sqrt[4]{\frac{8\mu L Q}{\pi \Delta p}} = \sqrt[4]{\frac{8\nu L w}{\pi \Delta p}}$$

in which $\nu = \mu/\rho$ and $w = \rho Q$. Using mks units we substitute into this formula to get:

$$R = \sqrt[4]{\frac{8(4.03 \times 10^{-5})(0.5002)(2.997 \times 10^{-3})}{(3.1416)(4.829 \times 10^{5})}}$$

$$= \sqrt[4]{3.185 \times 10^{-13}} = \underline{7.51 \times 10^{-4}\ \text{m}} \quad \text{or} \quad \underline{0.751\ \text{mm.}}$$

The Reynolds number for the system is: (mks units used here)

$$Re = \frac{D\langle v\rangle\rho}{\mu} = \frac{2}{\pi}\left(\frac{w}{R\nu\rho}\right) = \frac{2}{\pi}\frac{(2.997 \times 10^{-3})}{(7.512 \times 10^{-4})(4.03 \times 10^{-5})(0.9552 \times 10^{3})}$$

Hence $Re = 66.0$, and the flow is laminar (this justifies use of the Hagen-Poiseuille law. Inasmuch as $L_e = 0.035\ D\,Re = 0.35$ cm., allowance for the end effects would not change R by more than a factor of:

$$\sqrt[4]{\frac{L-L_e}{L}} = 0.998$$

2.B Volume Rate of Flow through an Annulus

We use Eq. 2.4-16 in which

$$\kappa = \frac{0.495}{1.1} = 0.45$$

$$\mu = (136.8\ \mathrm{lb_m\ ft^{-1}\,hr^{-1}})\left(\frac{1}{3600}\ \frac{\mathrm{hr}}{\mathrm{sec}}\right) = 3.80\times 10^{-2}\ \mathrm{lb_m\ ft^{-1}\,sec^{-1}}$$

$$(\mathcal{P}_0 - \mathcal{P}_L) = (5.37\ \mathrm{lb_f\ in^{-2}})(32.17\ \mathrm{poundals\ lb_f^{-1}})(144\ \mathrm{in^2\ ft^{-2}}) = 2.44\times 10^4\ \mathrm{poundals\ ft^{-2}}$$

$$R = (1.1\ \mathrm{in})\left(\tfrac{1}{12}\ \mathrm{ft\ in^{-1}}\right) = 0.0917\ \mathrm{ft}$$

Then substitution into Eq. 2.4-16 gives:

$$Q = \frac{\pi\,(2.44\times10^4\ \mathrm{lb_m\ ft^{-1}\,sec^{-2}})(0.0917\ \mathrm{ft})^4\ (0.163)}{8\ (3.8\times10^{-2}\ \mathrm{lb_m\ ft^{-1}\,sec^{-1}})\ (27\ \mathrm{ft})} = 0.108\ \mathrm{ft^3\,sec^{-1}}$$

To verify that it was proper to use the laminar flow formula of Eq. 2.4-16 we compute the Reynolds number:

$$Re = \frac{2R\,(1-\kappa)\,\langle v_z\rangle\rho}{\mu} = \frac{2}{\pi}\left(\frac{Q\rho}{R\mu}\right)\left(\frac{1}{1+\kappa}\right)$$

$$= \frac{2}{\pi}\,\frac{(0.108)(80.3)}{(0.0917)(3.8\times10^{-2})(1.45)} = 1090$$

2.C Loss of Catalyst Particles in a Stack Gas

(a) Rearrangement of Eq. 2.6-15 gives:

$$v_t = D^2(\rho_s - \rho)\,g\,/\,18\,\mu$$

in which D is the sphere diameter. When v_t is larger than 1.0 ft sec^{-1} (vertically downward) then the particle will not go up the stack. Hence we want to find that value of D for which $v_t = 1.0$ ft sec^{-1}. This will be the maximum diameter of particles that can be lost.

Using cgs units we get:

$$v_t = (1 \text{ ft sec}^{-1})(12 \text{ in ft}^{-1})(2.54 \text{ cm in}^{-1}) = 30.5 \text{ cm sec}^{-1}$$

$$\rho = (0.045 \text{ lb}_m \text{ ft}^{-3})(454 \text{ g lb}_m^{-1})(12 \times 2.54)^{-3} (\text{ft}^3 \text{ cm}^{-3})$$

$$= 7.2 \times 10^{-4} \text{ g cm}^{-3}$$

Therefore:

$$D_{max} = \sqrt{\frac{18 \mu v_t}{(\rho_s - \rho) g}} = \sqrt{\frac{(18)(0.00026)(30.5)}{(1.2 - 7.2\times10^{-4})(981)}}$$

$$= \sqrt{1.21 \times 10^{-4}} = 1.1 \times 10^{-2} \text{ cm} = 110 \text{ microns}$$

(b) Eq. 2.6–16 is of course valid only for $Re < 0.1$ although it may be applied approximately up to about $Re = 1$. For the system at hand

$$Re = \frac{D v_t \rho}{\mu} = \frac{(1.1 \times 10^{-2})(30.5)(7.2 \times 10^{-4})}{(0.00026)} = 0.93$$

In Chapter 6, methods are given for handling flow around spheres for $Re > 1$.

2.D Flow of Falling Film--- Alternate Derivations

(a) Set up a momentum balance as before and obtain the differential equation:

$$\frac{d\tau_{\bar{x}z}}{d\bar{x}} = \rho g \cos\beta$$

which may be integrated to give:

$$\tau_{\bar{x}z} = \rho g \bar{x} \cos\beta + C_1$$

Inasmuch as no momentum is transferred at $\bar{x} = \delta$, then at that plane we have $\tau_{\bar{x}z} = 0$. This boundary condition enables us to determine C_1 to be:

$$C_1 = -\rho g \delta \cos\beta$$

and hence the momentum flux distribution is:

$$\tau_{\bar{x}z} = -\rho g \delta \cos\beta \left[1 - (\bar{x}/\delta)\right]$$

Note that the momentum flux is in the negative $\bar{x}$-direction.

Insertion of Newton's law of viscosity $\tau_{\bar{x}z} = -\mu\,(dv_z/d\bar{x})$ into the above expression then gives the differential equation for the velocity distribution:

$$\frac{dv_z}{d\bar{x}} = \left(\frac{\rho g \delta}{\mu}\right) \cos\beta \left(1 - \frac{\bar{x}}{\delta}\right)$$

This first-order differential equation is easily integrated to give:

$$v_z = \left(\frac{\rho g \delta}{\mu}\right)^2 \cos\beta \left(\frac{\bar{x}}{\delta} - \frac{1}{2}\left(\frac{\bar{x}}{\delta}\right)^2\right)$$

the constant of integration, C_2, being zero because $v_z = 0$ at $\bar{x} = 0$.

Now we note that $\bar{x}$ and x are related thus:

$$\left(\frac{\bar{x}}{\delta}\right) = 1 - \left(\frac{x}{\delta}\right)$$

where x is the coordinate used in § 2.2. If the above relation is substituted into the velocity distribution we get:

$$v_z = \left(\frac{\rho g \delta}{\mu}\right)^2 \cos\beta \left[1 - \frac{x}{\delta} - \frac{1}{2}\left\{1 - 2\frac{x}{\delta} + \left(\frac{x}{\delta}\right)^2\right\}\right]$$

which, upon simplification, becomes:

$$v_z = \left(\frac{\rho g \delta}{2\mu}\right)^2 \cos\beta \left[1 - \left(\frac{x}{\delta}\right)^2\right]$$

which is the same as Eq. 2.2-16. This just illustrates that the choice of coordinate system makes no difference in the final answer.

(b) Substitution of Eq. 2.2-12 into Eq. 2.2-8 gives <u>for constant μ</u>

$$\frac{d^2 v_z}{dx^2} = \left(\frac{\rho g}{\mu}\right) \cos\beta$$

Integration twice gives:

$$v_z = \left(\frac{\rho g}{\mu}\right) \cos\beta \quad \frac{1}{2}x^2 + C_1 x + C_2$$

Application of the boundary conditions

B.C. 1 At $x = 0$, $dv_z/dx = 0$
B.C 2 At $x = \delta$, $v_z = 0$

gives two simultaneous equations for the integration constants C_1 and C_2:

$$\begin{cases} 0 = (\rho g/\mu)(\cos\beta)\cdot 0 + C_1 \\ 0 = (\rho g/\mu)(\cos\beta)\cdot \frac{1}{2}\delta^2 + C_1\delta + C_2 \end{cases}$$

whence $C_1 = 0$ and $C_2 = \frac{1}{2}(\rho g\delta^2/\mu)\cos\beta$. Insertion of these values into the last expression for v_z then gives finally:

$$v_z = (\rho g\delta^2/2\mu)(\cos\beta)\left[1-(x/\delta)^2\right]$$

which is the same as Eq. 2.2-16.

2.E Laminar Flow in a Narrow Slit

A differential momentum balance leads to:

$$\frac{d\tau_{xz}}{dx} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L}$$

Then, insertion of Newton's law gives

$$\frac{d^2 v_z}{dx^2} = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{\mu L}$$

Two successive integrations then yield:

$$v_z = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{2\mu L}x^2 + C_1 x + C_2$$

Use of the boundary conditions that $v_z = 0$ at $x = \pm B$ then gives the values of C_1 and C_2; when these are inserted into the last equation we get for the velocity distribution:

$$v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right]$$

Once the velocity profiles are known we can get various derived quantities:

The momentum flux distribution is: $\tau_{xz} = -\mu\frac{dv_z}{dx} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L}x$

The average velocity is:

$$\langle v_z\rangle = \frac{\int_0^W\int_0^B v_z\,dx\,dz}{\int_0^W\int_0^B dx\,dz} = \frac{1}{B}\int_0^B v_z\,dx$$

$$= \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu L}\int_0^1\left[1-\left(\frac{x}{B}\right)^2\right]d\left(\frac{x}{B}\right)$$

$$= (\mathcal{P}_0 - \mathcal{P}_L)B^2/3\mu L$$

The maximum velocity is: $$v_{z,max} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)B^2}{2\mu L}$$

Ratio of avg. to max. velocity: $$\langle v_z \rangle / v_{z,max} = \frac{2}{3}$$

Volume rate of flow: $$Q = BW\langle v_z \rangle = \frac{2}{3}\frac{(\mathcal{P}_0 - \mathcal{P}_L)B^3 W}{\mu L}$$

The latter result is the analog of the Hagen-Poiseuille law.

2.F Interrelation of Slit and Annulus Formulas

Substitution of $\kappa = 1 - \epsilon$ into Eq. 2.4-16 gives:

$$Q = \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R^4}{8\mu L}\left[(1 - 1 + 4\epsilon - 6\epsilon^2 + 4\epsilon^3 - \epsilon^4) + \frac{(1 - 1 + 2\epsilon - \epsilon^2)^2}{-\epsilon - \frac{1}{2}\epsilon^2 - \frac{1}{3}\epsilon^3 - \frac{1}{4}\epsilon^4 - \cdots}\right]$$

$$= \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R^4}{8\mu L}\left[4\epsilon - 6\epsilon^2 + 4\epsilon^3 - \epsilon^4 - \frac{4\epsilon^2 - 4\epsilon^3 + \epsilon^4}{\epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \cdots}\right]$$

When the indicated long division is performed we then get:

$$Q = \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R^4}{8\mu L}\left[\cancel{4\epsilon} - \cancel{6\epsilon^2} + 4\epsilon^3 - \epsilon^4 - \cancel{4\epsilon} + \cancel{6\epsilon^2} - \frac{8}{3}\epsilon^3 - \frac{2}{3}\epsilon^4 + \cdots\right]$$

or

$$Q = \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R^4\epsilon^3}{6\mu L}\left[1 - \frac{5}{4}\epsilon + \cdots\right]$$

If $\epsilon \ll 1$, then the term $\frac{5}{4}\epsilon$ and higher terms may be neglected.

2.G Laminar Flow of a Falling Film on Outside of a Circular Tube

(a) We set up a momentum balance over a thin shell of thickness Δr:

$$\{\text{MOMENTUM IN}\} \; - \; \{\text{MOMENTUM OUT}\} \; + \; \{\text{FORCE}\} = 0$$

$$(2\pi r L \tau_{rz})\Big|_{r} - (2\pi r L \tau_{rz})\Big|_{r+\Delta r} + 2\pi L \Delta r \rho g = 0$$

Divide by $2\pi L \Delta r$ and take the limit as $\Delta r \to 0$

$$-\frac{d}{dr}\left(r\tau_{rz}\right) + r\rho g = 0$$

Substitution of Newton's law into this equation gives: (for constant μ)

$$+\mu \frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -\rho g r$$

Two integrations then give:

$$v_z = -\frac{\rho g r^2}{4\mu} + C_1 \ln r + C_2$$

Application of the boundary conditions:

B.C. 1: At $r = R$, $v_z = 0$

B.C. 2: At $r = aR$, $dv_z/dr = 0$

gives the following two equations for C_1 and C_2

B.C. 1: $$0 = -\frac{\rho g R^2}{4\mu} + C_1 \ln R + C_2$$

B.C. 2: $$0 = -\frac{\rho g aR}{2\mu} + \frac{C_1}{aR}$$

These may be solved for C_1 and C_2, and the results substituted into the expression for v_z above; hence we finally get:

$$v_z = \frac{\rho g R^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2 + 2a^2 \ln \frac{r}{R}\right]$$

(b) The volume rate of flow is:

$$Q = \int_0^{2\pi}\int_R^{aK} v_z \, r \, dr \, d\theta = 2\pi K^2 \int_1^{a} v_z \, \xi \, d\xi \qquad \text{where } \xi = \frac{r}{K}$$

Substitution of the velocity profile into this integral gives:

$$Q = \frac{\pi \rho g R^4}{2\mu} \int_1^a \left(\xi - \xi^3 + 2a^2 \xi \ln \xi \right) d\xi$$

$$= \frac{\pi \rho g R^4}{2\mu} \left(\frac{\xi^2}{2} - \frac{\xi^4}{4} + 2a^2 \left[-\frac{1}{4}\xi^2 + \frac{1}{2}\xi^2 \ln \xi \right] \right) \Bigg|_1^a$$

$$= \frac{\pi \rho g R^4}{8\mu} \left(-1 + 4a^2 - 3a^4 + 4a^4 \ln a \right)$$

(c) If we set $a = 1 + \epsilon$ (where ϵ is small) and expand in powers of ϵ, we get:

$$Q = \frac{\pi \rho g R^4}{8\mu} \left(\frac{16}{3} \epsilon^3 + \text{higher powers of } \epsilon \right)$$

or

$$Q \approx \frac{2\pi \rho g R^4 \epsilon^3}{3\mu}$$

This is in agreement with Eq. 2.2-19 with $W = 2\pi R$, $\delta = \epsilon R$, and $\cos\beta = 1$.

2.H Non-Newtonian Flow in a Tube

(a) Combination of Eq. 2.H-1 and Eq. 2.3-12 gives:

$$m \left(-\frac{dv_z}{dr} \right)^n = \frac{(\mathcal{P}_0 - \mathcal{P}_L) r}{2L}$$

Let $s = 1/n$; then:

$$-\frac{dv_z}{dr} = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{2mL} \right)^s r^s$$

Integration gives:

$$-v_z = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{2mL} \right)^s \left[\frac{r^{s+1}}{s+1} + C \right]$$

The boundary condition of zero velocity at the wall allows one to evaluate C, with the result that:

$$v_z = \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L) R}{2mL} \right)^s \frac{R}{s+1} \left[1 - \left(\frac{r}{R} \right)^{s+1} \right]$$

The volume rate of flow is then:

$$Q = 2\pi \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L) R}{2mL} \right)^s \frac{R^3}{s+1} \int_0^1 \left[1 - \left(\frac{r}{R} \right)^{s+1} \right] \left(\frac{r}{R} \right) d\left(\frac{r}{R} \right)$$

Hence, when the integration is performed:

$$Q = \pi \left[\frac{(\mathcal{P}_0 - \mathcal{P}_L) R}{2mL} \right]^s \frac{R^3}{s+3}$$

Note that when $s = 1$ and $m = \mu$, this becomes the Hagen-Poiseuille formula

(b) For the Ellis fluid the velocity distribution is:

$$v_z = \frac{\varphi_0 (\mathcal{P}_0 - \mathcal{P}_L) R^2}{4L} \left[1 - \left(\frac{r}{R} \right)^2 \right] + \varphi_1 \left[\frac{(\mathcal{P}_0 - \mathcal{P}_L) R}{2L} \right]^\alpha \frac{R}{\alpha+1} \left[1 - \left(\frac{r}{R} \right)^{\alpha+1} \right]$$

and the volume rate of flow is:

$$Q = \frac{\pi (\mathcal{P}_0 - \mathcal{P}_L) R^4 \varphi_0}{8L} + \pi \left[\frac{(\mathcal{P}_0 - \mathcal{P}_L) R}{2L} \right]^\alpha \frac{R^3 \varphi_1}{\alpha+3}$$

These results may be obtained by the same process described in (a).

2.I Flow of a Bingham Fluid from a Circular Tube

There will be flow only if the greatest value of the momentum flux τ_{rz} exceeds the value of τ_0 characterizing the Bingham fluid. The greatest value of τ_{rz} occurs at the wall and is $\rho g R/2$. Hence:

If $\rho g R/2 < \tau_0$ fluid will not flow

If $\rho g R/2 > \tau_0$ fluid will flow

2.J. Annular Flow with Inner Cylinder Moving Axially

A shell momentum balance leads to the differential equation

$$\frac{d}{dr}\left(r \frac{dv_z}{dr}\right) = 0$$

This has to be integrated with the boundary conditions

B.C. 1 : At $r = \kappa R$, $v_z = V$

B.C. 2 : At $r = R$, $v_z = 0$

When the integration is performed and the two integration constants are evaluated from the above two boundary conditions, we get:

$$\frac{v_z}{V} = \frac{\ln (r/R)}{\ln \kappa}$$

Then the volume rate of flow is found to be:

$$Q = 2\pi \int_{\kappa R}^{R} v_z \, r \, dr$$

$$= \frac{2\pi R^2 V}{\ln \kappa} \int_{\kappa}^{1} \ln\left(\frac{r}{R}\right) \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

$$= \frac{2\pi R^2 V}{\ln \kappa} \left[-\frac{1}{4}\xi^2 + \frac{1}{2}\xi^2 \ln \xi \right] \Big|_{\kappa}^{1}$$

$$= \frac{\pi R^2 V}{2} \left[\frac{(1-\kappa^2)}{\ln (1/\kappa)} - 2\kappa^2 \right]$$

2.K Non-Newtonian Film Flow

According to §2.2 for any fluid $\tau_{xz} = \rho g x$. When the rheological equation for the Bingham fluid is inserted into this, we get:

$$\tau_0 - \mu_0 \frac{dv_z}{dx} = \rho g x$$

When this is integrated we get:

$$v_z = -\frac{\rho g}{2\mu_0} x^2 + \frac{\tau_0}{\mu_0} x + C_2$$

At $x = \delta$, $v_z = 0$ so that

$$0 = -\frac{\rho g}{2\mu_0}\delta^2 + \frac{\tau_0}{\mu}\delta + C_2$$

Subtracting these last two equations gives:

$$v_z = \frac{\rho g \delta^2}{2\mu_0}\left[1 - \left(\frac{x}{\delta}\right)^2\right] - \frac{\tau_0 \delta}{\mu_0}\left[1 - \left(\frac{x}{\delta}\right)\right] \qquad (x_0 \leq x \leq \delta)$$

for the velocity distribution in the range $x_0 \leq x \leq \delta$. The velocity in the "plug flow region" $(0 \leq x \leq x_0)$ is:

$$v_z = \frac{\rho g \delta^2}{2\mu_0}\left[1 - \frac{x_0}{\delta}\right]^2 \qquad (0 \leq x \leq x_0)$$

where $\tau_0 = \rho g x_0$ is the defining equation for x_0.

Next, we get the mass rate of flow (with W = width of film, Q = volume rate of flow):

$$\Gamma = \frac{\rho Q}{W} = \rho \int_0^\delta v_z \, dx$$

Integration by parts

$$= \rho\left[v_z x \Big|_0^\delta - \int_0^\delta x\left(\frac{dv_z}{dx}\right) dx\right]$$

$$= \rho \int_{x_0}^\delta x \left(-\frac{dv_z}{dx}\right) dx$$

$$= \rho \int_{x_0}^\delta x \left[\frac{\rho g}{\mu_0}x - \frac{\tau_0}{\mu_0}\right] dx$$

Finally

$$\Gamma = \frac{\rho^2 g \delta^3}{3\mu_0}\left[1 - \frac{3}{2}\left(\frac{x_0}{\delta}\right) + \frac{1}{2}\left(\frac{x_0}{\delta}\right)^3\right]$$

Hence δ would have to be obtained from a graphical solution, by plotting

$$\frac{3\Gamma\mu_0}{\rho^2 g \delta^3} \quad \underline{\text{vs.}} \quad \frac{x_0}{\delta}$$

CHAPTER 3 — Checked by V. D. Shah

3.A Torque Required to Turn a Friction Bearing

In Eq. 3.5-13 is given an expression for calculating the torque needed to turn an outer rotating cylinder at an angular velocity Ω_o. For the problem described in Prob. 3.A we need a similar formula for the torque need to turn an inner rotating cylinder at an angular velocity Ω_i. It should be obvious that the two expressions must be the same. If it is not one can easily show that when the inner cylinder is rotated at Ω_i and the outer held stationary, then the velocity distribution is *

$$v_\theta = \frac{\kappa R \Omega_i}{\left(\frac{1}{\kappa} - \kappa\right)} \left[\left(\frac{R}{r}\right) - \left(\frac{r}{R}\right) \right]$$

and the torque is then:

$$\mathcal{T} = (2\pi\kappa R L)\left(+ \tau_{r\theta}\Big|_{r=\kappa R} \right)(\kappa R)$$

or

$$\mathcal{T} = 4\pi L \mu \Omega_i R^2 \left(\frac{\kappa^2}{1-\kappa^2}\right)$$

which is the same in form as Eq. 3.5-13

For the situation described in Problem 3.A we evaluate the various quantities needed:

$$\kappa = \frac{\text{inner radius}}{\text{outer radius}} = \frac{1.000''}{1.002''} = 0.998 \;; \qquad \kappa^2 = 0.996$$

$$\left(\frac{\kappa^2}{1-\kappa^2}\right) = \frac{0.996}{0.004} = 249$$

These conversion factors come from Table C.3-1

$$\mu = 200 \text{ cp} = (200)(6.72\times10^{-4}) \text{ lb}_m \text{ ft}^{-1} \text{ sec}^{-1} = 0.1344 \text{ lb}_m \text{ ft}^{-1} \text{ sec}^{-1}$$

$$= (200)(2.09\times10^{-5}) \text{ lb}_f \text{ sec ft}^{-2} = 4.18\times10^{-3} \text{ lb}_f \text{ sec ft}^{-2}$$

$$\Omega_i = (200 \text{ rpm})\left(\frac{1}{60}\frac{\text{min}}{\text{sec}}\right)\left(2\pi \frac{\text{radians}}{\text{revolution}}\right) = \frac{20}{3}\pi \frac{\text{radians}}{\text{sec}}$$

$$R^2 = (1 \text{ in})^2 = \left(\frac{1}{12}\right)^2 \text{ft}^2 = \frac{1}{144} \text{ ft}^2$$

$$L = 2 \text{ in} = \frac{1}{6} \text{ ft}$$

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* See Problem 3.F

Hence the torque is

$$\mathcal{T} = (4\pi)\left(\frac{1}{6}\right)\left(4.18\times10^{-3}\right)\left(\frac{20\pi}{3}\right)\left(\frac{1}{144}\right)(249)$$
$$= 0.32 \ lb_f \, ft$$

And the power is

$$P = \mathcal{T}\Omega = (0.32 \ lb_f \, ft)\left(\frac{20}{3}\pi \ sec^{-1}\right)\left(\frac{1}{550} \frac{hp}{ft \, lb_f \, sec^{-1}}\right)$$
$$= 0.012 \ hp$$

In using the above formulae it has been tacitly assumed that the flow is stable and laminar. We must verify that this is indeed so by using Eq. 3.5-14. Hence we must check to be sure that:

$$\left(\frac{\Omega_i \kappa R^2 \rho}{\mu}\right)(1-\kappa)^{3/2} < 41.3$$

Inserting the values from this example we have:

$$\frac{\left(\frac{20\pi}{3}\right)(0.998)\left(\frac{1}{144}\right)(50)}{(0.1344)} (0.002)^{3/2} \approx 10^{-2}$$

which is well below the critical limit of 41.3.

3.B. The Cone-and-Plate Viscometer

Throughout this problem we recognize that in the conical slit $\theta \approx \pi/2$ and that $\sin\theta$ can everywhere be set equal to unity to a very good approximation; hence the expressions for torque, momentum flux, and velocity profile may be written as follows:

From Eq. 3.5-32: $\quad \mathcal{T} \doteq \frac{2}{3}\pi R^3 \tau_{\theta\phi}$

From Eq. 3.5-33: $\quad \tau_{\theta\phi} \doteq -\mu \frac{d}{d\theta}\left(\frac{v_\phi}{r}\right)$

From Eq. 3.5-37: $\quad \frac{v_\phi}{r} \doteq \Omega \frac{\left(\frac{\pi}{2}-\theta\right)}{\left(\frac{\pi}{2}-\theta_1\right)}$

When these three relations are combined we get

$$\mathcal{T} \doteq \frac{2}{3}\pi R^3 \mu \Omega \Big/ \left(\frac{\pi}{2} - \theta_1\right)$$

which is a very good approximation for very tiny cone angles. [This same result is given by S. Oka in Vol. III of Eirich's "Rheology" (Academic Press, 1960)— see p. 62.]

For the situation at hand

$$R = 10 \text{ cm}$$

$$\mu = 100 \text{ cp} = 1 \text{ g cm}^{-1} \text{sec}^{-1}$$

$$\Omega = 10 \text{ radians min}^{-1} = \frac{10}{60} \frac{\text{radians}}{\text{sec}} = \frac{1}{6} \frac{\text{radians}}{\text{sec}}$$

$$\frac{\pi}{2} - \theta_1 = 0.5° = \frac{\pi}{360} \text{ radians}$$

Hence the torque is:

$$\mathcal{T} \doteq \frac{2}{3}\pi (10)^3 (1)\left(\frac{1}{6}\right) \Big/ \frac{\pi}{360} = 4 \times 10^4 \text{ dyn-cm}$$

3.C The Effect of Altitude on Air Pressure

If we assume a stationary atmosphere — i.e., no wind currents — then the equation of motion is:

$$\frac{dp}{dz} = \rho g \qquad \text{(z is measured from level of Lake Superior)}$$

If it be assumed that the ideal gas law is applicable, then

$$\rho = \frac{pM}{RT}$$

From the given temperature data the temperature in °R is:

$$T(z) = 530 - (3 \times 10^{-3}) z$$

Then the equation of motion may be integrated:

$$\int_{750}^{p} dp = \int_{0}^{1421} \rho g \, dz = \frac{Mg}{R} \int_{0}^{1421} \frac{dz}{530 - (3 \times 10^{-3}) z}$$

When the integrations are performed, we get: