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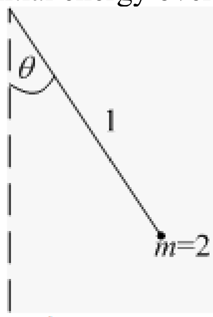
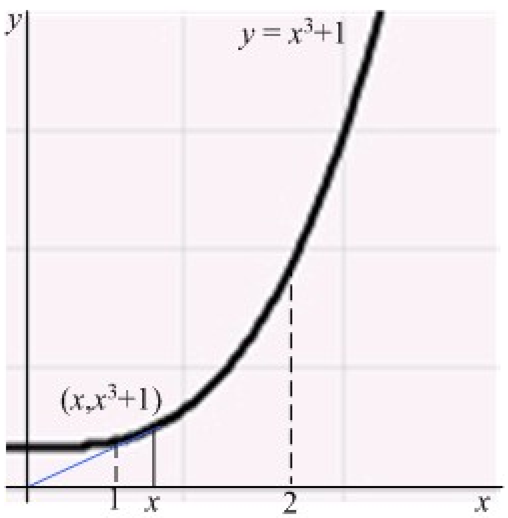
Chapter 02

1. The potential energy V of a pendulum of length 1 and mass 2, relative to its rest position is $V = 2g(1 - \cos\theta)$. Compute the average rate of change of the potential energy over the angle interval $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$.



ANSWER: $\frac{12g\left(\sqrt{2}-1\right)}{\pi} \approx 1.58g$

2. Let $S(x)$ denote the slope of the line segment connecting the origin to the point (x, y) on the graph of the equation $y = x^3 + 1$. Calculate the average rate of change of $S(x)$ for $1 \le x \le 2.$



ANSWER: $2\frac{1}{2}$

3. The flight time of a shell shot at an angle θ and initial velocity V is $T_f = \frac{2V}{g}\sin\theta$. Compute the average rate of change of the flight time for θ in the interval $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$.

ANSWER: $\frac{12\left(\sqrt{2}-1\right)V}{\pi g}$

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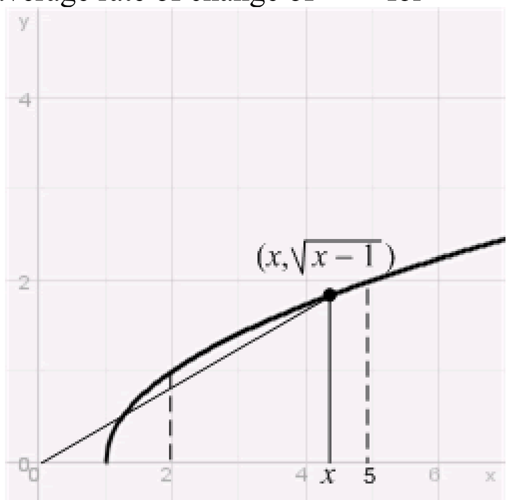
4. Let $S(x)$ denote the slope of the line segment connecting the origin to the point (x, y) on the graph of the equation $y = \sqrt{x-1}$. Calculate the average rate of change of $S(x)$ for $2 \le x \le 5$.



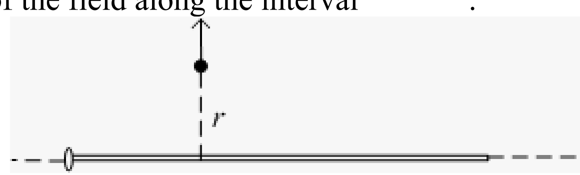
ANSWER:

$-\frac{1}{30}$

5. The volume of a cone of radius R and height H is $V = \frac{\pi R^2 H}{3}$.
What is the average rate of change of V if the radius increases from 1 to 3 and the height remains unchanged?
ANSWER:

$\frac{4\pi H}{3}$

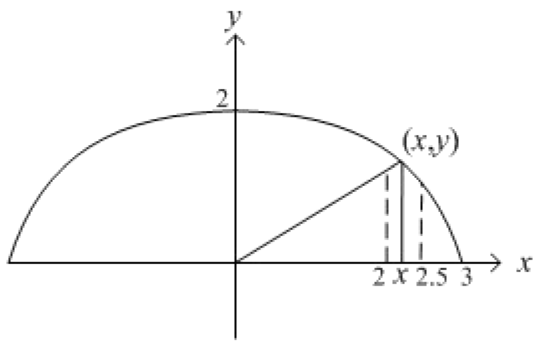
6. The electrical field due to an infinite rod at a point at distance r from the rod is perpendicular to the rod and has a magnitude of $E(r) = \frac{1}{2\pi\varepsilon_0} \cdot \frac{\lambda}{r}$ (ε_0 is a constant and λ is the longitudinal charge density).
Find the average rate of change of the field along the interval $2 \le r \le 4$.



ANSWER:

$-\frac{\lambda}{16\pi\varepsilon_0}$

7. Let $S(x)$ denote the slope of the line segment connecting the origin to the point (x, y) on the graph of the semi-ellipse $y = 2\sqrt{1 - \frac{x^2}{9}}$. Calculate the average rate of change of $S(x)$ for $2 \le x \le 2.5$



ANSWER: -0.206 per unit length

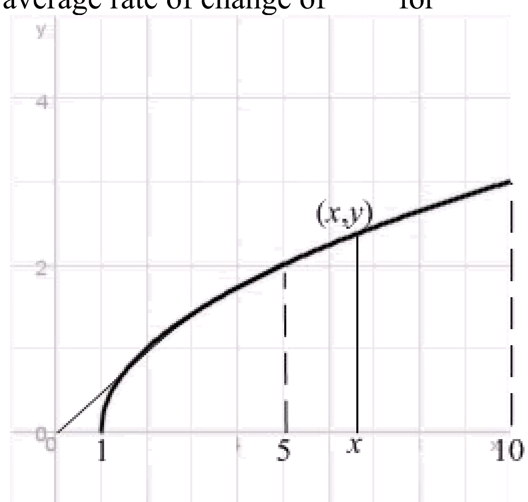
8. The electrical field caused by an electrical charge q at a point at distance r is $E = \frac{kq}{r^2}$ (k is a constant). Find the average rate of change of the field along the interval $1 \le r \le 3$.

ANSWER: $-\frac{4kq}{9}$

9. The volume of a sphere of radius R is $V = \frac{4\pi R^3}{3}$. What is the average rate of change of the volume when the radius increases from $R = 1$ to $R = 3$?

ANSWER: $\frac{52\pi}{3}$

10. Let $S(x)$ denote the slope of the line segment connecting the origin to the point (x, y) on the graph of the equation $y = \sqrt{x-1}$. Calculate the average rate of change of $S(x)$ for $5 \le x \le 10.$



ANSWER: $-\frac{1}{50}$

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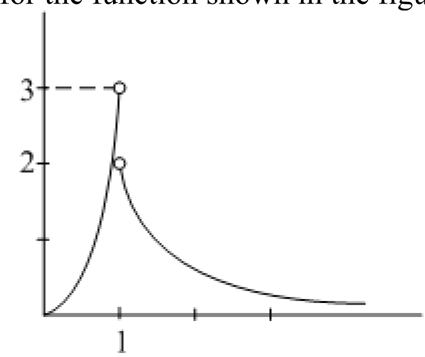
11. The position of a particle is given by $g(t) = 2t^2 + 5$. Compute the average velocity over the time interval $[4, 6]$. Estimate the instantaneous velocity at $t = 4$.

ANSWER: Average velocity over $[4, 6]$: 20
Instantaneous velocity at $t = 4$: 16

12. A balloon is blown up and takes the shape of a sphere. What is the average rate of change of the surface area of the balloon as the radius increases from 3 to 4 cm?

ANSWER: 28π

13. Determine $\lim_{x \to 1+} f(x)$ and $\lim_{x \to 1-} f(x)$ for the function shown in the figure.



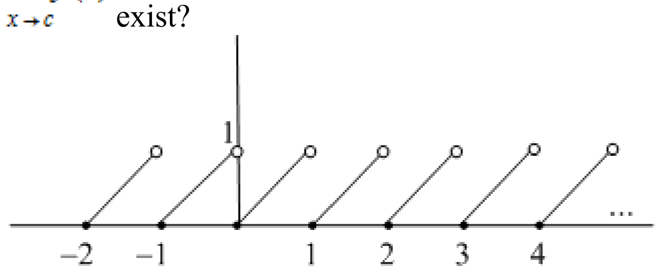
ANSWER: $\lim_{x \to 1+} f(x) = 2$ $\lim_{x \to 1-} f(x) = 3$

14. The greatest integer function is defined by $[x] = n$, where n is the unique integer such that $n \le x < n + 1$. The graph of $f(x) = x - [x]$ is shown in the figure.

A) For which values of c does $\lim_{x \to c-} f(x)$ exist?

B) For which values of c does $\lim_{x \to c+} f(x)$ exist?

C) For which values of c does $\lim_{x \to c} f(x)$ exist?



ANSWER: A) All *c*
B) All *c*
C) Every real number *c* that is not an integer

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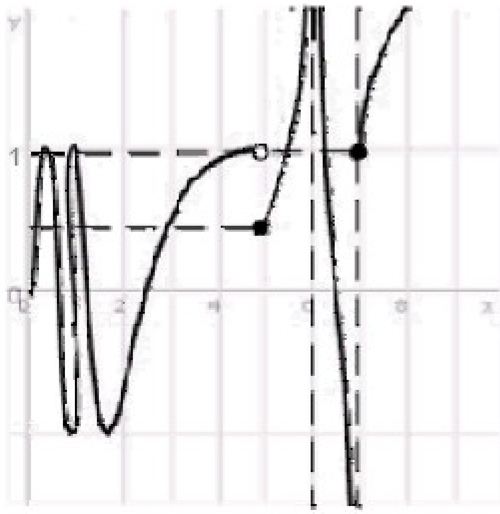
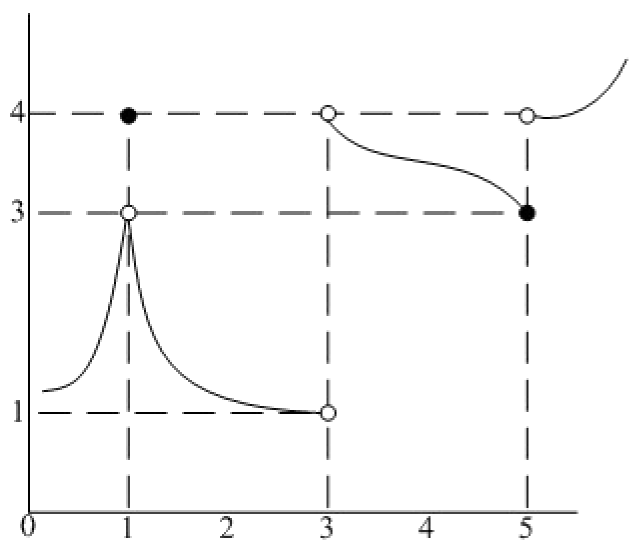
15. The graph of a function $y = f(x)$ is shown in the figure.
Determine the following limits or state that the limit does not exist (if the limit is infinite, write ∞ or $-\infty$):

A) $\lim_{x \to 0+} f(x)$ B) $\lim_{x \to 5-} f(x)$ C) $\lim_{x \to 5+} f(x)$ D) $\lim_{x \to 5} f(x)$ E) $\lim_{x \to 6-} f(x)$
F) $\lim_{x \to 6+} f(x)$ G) $\lim_{x \to 6} f(x)$ H) $\lim_{x \to 7-} f(x)$ I) $\lim_{x \to 7+} f(x)$ J) $\lim_{x \to 7} f(x)$



ANSWER: A)0 B)1 C)$\frac{1}{2}$ D)Does not exist E) ∞
F) ∞ G) ∞ H) $-\infty$ I)1 J)Does not exist

16. Determine the one-sided limits at $c = 1, 3, 5$ of the function $f(x)$ shown in the figure and state whether the limit exists at these points.



ANSWER: $\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = 3$; limit exists

$\lim_{x \to 3-} f(x) = 1, \lim_{x \to 3+} f(x) = 4$; limit does not exist

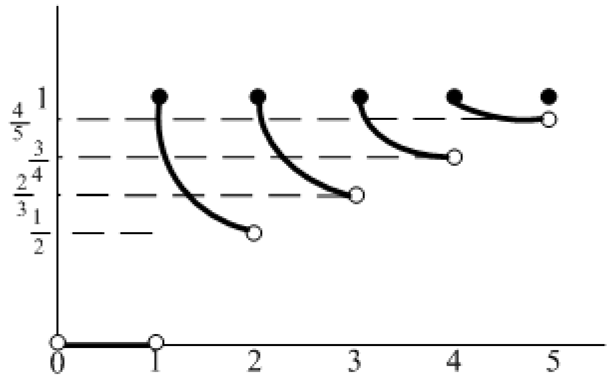
$\lim_{x \to 5-} f(x) = 3, \lim_{x \to 5+} f(x) = 4$; limit does not exist

17. Consider the function $f(x) = \frac{[x]}{x}$ for $x > 0$. (Here, [*x*] denotes the greatest integer function.)

A) Write $f(x)$ in piecewise form.
What is $f(n)$ for positive integers n?

B) Determine $\lim_{x \to 3-} f(x)$ and $\lim_{x \to 3+} f(x)$.

C) For which values of c does $\lim_{x \to c} f(x)$ exist?



ANSWER:

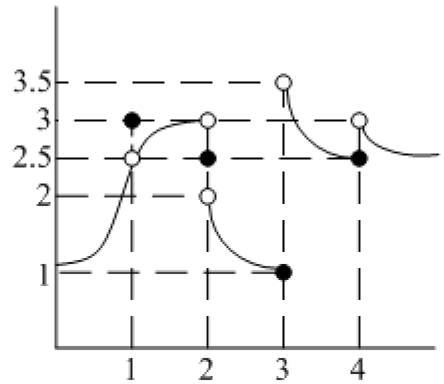
$$f(x) = \begin{cases} 0 & 0 < x < 1 \\ \frac{n}{x} & n \le x < n+1 \quad n = 1, 2, \ldots \end{cases}$$

A) $f(n) = 1 \quad n = 1, 2, \ldots$

B) $\frac{2}{3}, 1$

C) The limit exists for all positive real numbers that are not integers.

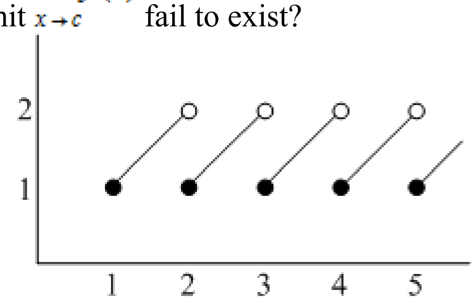
18. Determine the one-sided limits at $c = 1, 2, 3, 4$ of the function shown in the figure and state whether the limit exists at these points.



ANSWER:

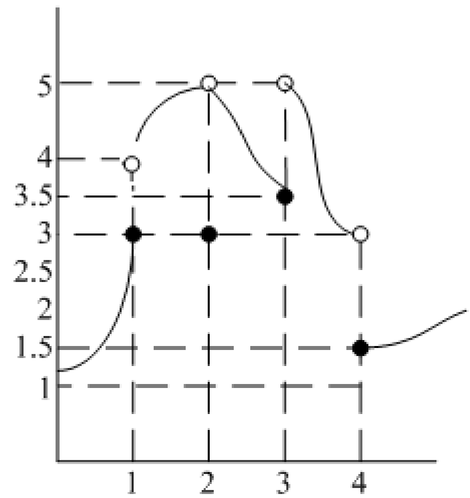
c	Left-sided	Right-sided	Limit
1	2.5	2.5	Exists (2.5)
2	3	2	Does not exist
3	1	3.5	Does not exist
4	2.5	3	Does not exist

19. Consider the function $f(x) = x + 1 - [x]$ for $x \geq 1$. (Here, [x] denotes the greatest integer function.)
A) Write f in piecewise form. What is $f(n)$ for positive integers $n \geq 1$?
B) Find $\lim_{x \to 2-} f(x)$ and $\lim_{x \to 2+} f(x)$.
C) For which values of c does the limit $\lim_{x \to c} f(x)$ fail to exist?



ANSWER: A) $f(x) = x + 1 - n$ for $n \leq x < n + 1$, $f(n) = 1$
B) $2, 1$
C) The limit fails to exist for all positive integers.

20. Determine the one-sided limits at $c = 1, 2, 3, 4$ of the function shown in the figure and state whether the limit exists at these points.



ANSWER: c	Left-sided	Right-sided	Limit
1	3	4	Does not exist
2	5	5	Exists (5)
3	3.5	5	Does not exist
4	3	1.5	Does not exist

21. Let $f(x)$ be the following function defined for $-0.5 \le x \le 4.5$:

$$f(x) = \begin{cases} 1, \text{ if } \sin\left(\frac{\pi x}{2}\right) > 0 \\ -1, \text{ if } \sin\left(\frac{\pi x}{2}\right) < 0 \\ 0, \text{ if } \sin\left(\frac{\pi x}{2}\right) = 0 \end{cases}$$

Write $f(x)$ as a piecewise-defined function where the intervals are in terms of x instead of $\sin\left(\frac{\pi x}{2}\right)$, sketch its graph, and determine the points where the limit of $f(x)$ does not exist. Find the one-sided limits at these points.

ANSWER:

$$f(x) = \begin{cases} -1 & -0.5 \le x < 0 \\ 0 & x = 0 \\ 1 & 0 < x < 2 \\ 0 & x = 2 \\ -1 & 2 < x < 4 \\ 0 & x = 4 \\ 1 & 4 < x \le 4.5 \end{cases}$$

$x = -0.5, 0, 2, 4, 4.5$

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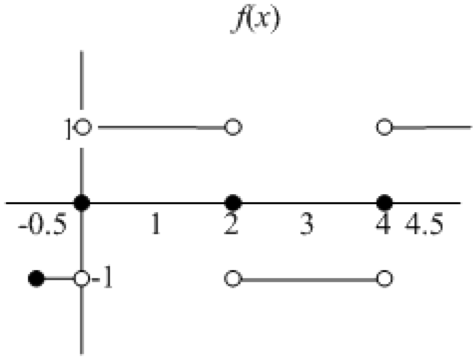
$\lim_{x \to -0.5+} f(x) = -1$ $\lim_{x \to 0-} f(x) = -1$ $\lim_{x \to 0+} f(x) = 1$ $\lim_{x \to 2-} f(x) = 1$

$\lim_{x \to 2+} f(x) = -1$ $\lim_{x \to 4-} f(x) = -1$ $\lim_{x \to 4+} f(x) = 1$ $\lim_{x \to 4.5-} f(x) = 1$



22. Find a real number c such that $\lim_{x \to 1} f(x)$ exists and compute the limit.

$$f(x) = \begin{cases} x - \dfrac{3}{x-2} & x < 1 \\ 10 & x = 1 \\ \dfrac{c}{(x+1)^2} & x > 1 \end{cases}$$

ANSWER: $c = 16;\ \lim_{x \to 1} f(x) = 4$

23. Let $\lim_{x \to a} f(x) = L$. Determine whether each of the following statements is always true, never true, or sometimes true.

A) $\lim_{x \to a-} f(x) = L$

B) $4f(a) = 3L$

C) $\lim_{x \to a-} f(x) - \lim_{x \to a+} f(x) < 0$

D) $\dfrac{\lim_{x \to a-} f(x)}{\lim_{x \to a+} f(x)} = 1$

ANSWER: A) Always
B) Sometimes
C) Never

D) Sometimes (note the case when $\lim_{x \to a} f(x) = 0$)

24. Compute the following one-sided limits:

A) $\lim_{x \to 2-} \frac{\sqrt{2-x}}{x^2+5x}$

B) $\lim_{\theta \to 0+} \frac{\theta^3 \cos^2 \theta}{\sin \theta}$

C) $\lim_{\theta \to \frac{\pi}{2}+} \frac{\theta^3}{\tan \theta}$

ANSWER:

A) 0
B) 0
C) 0

25. Evaluate the limits using the Limit Laws:

A) $\lim_{x \to -1} \left(x^3 - 2x^2 + 1\right)$

B) $\lim_{t \to 1} \frac{t^2 - t}{t+1}$

C) $\lim_{x \to 0} \frac{1 + \cos x}{x^3 + 2}$

D) $\lim_{t \to 0} \frac{3 \sin t}{2t}$

ANSWER:

A) -2
B) 0
C) 1
D) $\frac{3}{2}$

26. Which of the following functions are examples of the existence of the limit $\lim_{x \to 0} \frac{f(x)}{g(x)}$, although the limits of $f(x)$ and $g(x)$ as $x \to 0$ do not exist?

a. $f(x) = x \;\; g(x) = \frac{1}{x}$

b. $f(x) = \frac{\sin x}{x} \;\; g(x) = \frac{1}{x}$

c. $f(x)=\frac{1}{x}$ $g(x)=\frac{1}{x^3}$

d. $f(x)=x^2$ $g(x)=\cos x$

e. $f(x)=\frac{x}{\sin x}$ $g(x)=\frac{1}{x}$

ANSWER: c

27. Let $f(x)$, $g(x)$ be functions and let $F(x)=f(x)-g(x)$. Consider the following statement: If $\lim_{x\to x_0} F(x)$ and $\lim_{x\to x_0} g(x)$ exist, then $\lim_{x\to x_0} f(x)$ also exists. To prove this statement, we should use which of the following?

a. The statement is not true.

b. The Product Rule applied to $\frac{F+g}{g}$ and g.

c. The Quotient Rule applied to $(F+g)g$ and g.

d. The Sum Rule applied to F and g.

ANSWER: d

28. Evaluate the limits using the Limit Laws:

A) $\lim_{t\to(-2)} (2t+1)(t^2+2)$

B) $\lim_{x\to(-1)} \frac{x^2+3x}{x-1}$

C) $\lim_{x\to\frac{\pi}{4}} \frac{\sin x+\cos x}{2\tan x}$

D) $\lim_{x\to 4} \frac{2x^{-1}+x^{-\frac{1}{2}}}{x+3}$

ANSWER:

A) -18

B) 1

C) $\frac{\sqrt{2}}{2}$

D) $\frac{1}{7}$

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29. Determine whether the following statement is correct: If $\lim_{x\to 0} xg(x)=0$, then $\lim_{x\to 0} g(x)$ exists. If yes, prove it; otherwise, give a counter example

ANSWER:

False; $g(x)=\begin{cases} 1 & x\ge 0 \\ -1 & x<0 \end{cases}$

30. Evaluate the limits using the Limit Laws:

A) $\lim_{t\to 3}\left(t^2+t-1\right)\sin\frac{\pi t}{2}$

B) $\lim_{x\to -1}\frac{x^3+5}{x^2+2x-1}$

C) $\lim_{y\to 4}\frac{y^{-\frac{1}{2}}\tan\left(\frac{\pi y}{16}\right)}{\sqrt{y^2+9}}$

ANSWER:

A) -11
B) -2
C) $\frac{1}{10}$

31. A) Can the Product Rule be used to compute the limit $\lim_{x\to 0}[x]x$? (Here, $[x]$ denotes the greatest integer function.) Explain.

B) Show that $\lim_{x\to 0}[x]x$ exists and find it. *Hint:* Compute the one-sided limits.

ANSWER:

A) No. The limit $\lim_{x\to 0}[x]$ does not exist.
B) 0

32. Let $f(x)$, $g(x)$, and $F(x)=f(x)+g(x)$. To prove that if $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} F(x)$ exist then also $\lim_{x\to x_0} g(x)$ exists, we should use which of the following?

a. The Product Rule applied to $\frac{F-f}{f}$ and f.
b. The Quotient Rule applied to $(F-f)f$ and f.
c. The Sum Rule applied to F and $-f$.
d. The statement is not true.
e. Both A and C

ANSWER: c

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33. Evaluate the limits using the Limit Laws:

A) $\lim_{x \to -1} \frac{x+2}{x^3 - x - 1}$

B) $\lim_{x \to 1} (x^2 - x^{-3} + x)(x+1)$

C) $\lim_{x \to \frac{\pi}{4}} \frac{\sin x + \cos x}{x}$

ANSWER:

A) -1
B) 2
C) $\frac{4\sqrt{2}}{\pi}$

34. Consider this statement: If $\lim_{x \to x_0} f(x) = c \neq 0$ and $\lim_{x \to x_0} g(x) = 0$, then $\frac{f(x)}{g(x)}$ does not converge to a finite limit as $x \to x_0$.

To prove this statement, we assume that $\lim_{x \to x_0} \frac{f(x)}{g(x)} = M$ exists and is finite. Then, by the Quotient Rule, $\lim_{x \to x_0} \frac{g(x)}{f(x)} = 0$ and by the Product Rule, $\lim_{x \to x_0} \left(\frac{f(x)}{g(x)} \cdot \frac{g(x)}{f(x)} \right) = 0$.

Which of the following statements completes the proof?

a. From $\lim_{x \to x_0} \left(\frac{f(x)}{g(x)} \cdot \frac{g(x)}{f(x)} \right) = 0$, it follows that 1 = 0, which is a contradiction.

b. From $\lim_{x \to x_0} \frac{g(x)}{f(x)} = 0$, we can conclude that $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$, which contradicts our assumption.

c. From $\lim_{x \to x_0} \frac{g(x)}{f(x)} = 0$, we can conclude that $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \infty$, which contradicts our assumption.

d. From $\lim_{x \to x_0} \left(\frac{f(x)}{g(x)} \cdot \frac{g(x)}{f(x)} \right) = 0$, we can conclude that $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$, which contradicts our assumption.

ANSWER: a

35. Which of the following functions are examples of the existence of the limit $\lim_{x \to 0} \frac{f(x)}{g(x)}$, although the limits $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist?

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a. $f(x)=\frac{1}{x}$, $g(x)=x^2$

b. $f(x)=\frac{1}{x}$, $g(x)=\frac{1}{\sin x}$

c. $f(x)=[x]$, $g(x)=x$ (Here, $[x]$ denotes the greatest integer function.)

d. $f(x)=\frac{1}{x^2}$, $g(x)=\frac{1}{x}$

e. $f(x)=\frac{1}{1-x}$, $g(x)=\frac{1}{1-\sin x}$

ANSWER: b

36. Assume a and L are nonzero real numbers. If $\lim_{x\to a} 2f(x)=L$ and $\lim_{x\to a}\frac{g(x)}{4}=0$, calculate the following limits, if possible. If not, state that it is not possible.

A) $\lim_{x\to a} f(x)\cdot g(x)$

B) $\lim_{x\to a}\frac{f(x)}{g(x)}$

C) $\lim_{x\to a}\frac{f(x)+x^2}{g(x)+a}$

ANSWER:

A) 0

B) Not possible

C) $\frac{\frac{L}{2}+a^2}{a}$

37. Determine the points at which the following functions are not continuous and state the type of discontinuity: removable, jump, infinite, or none of these.

A) $f(x)=\frac{x^2-1}{|x-3|}$

B) $g(x)=\frac{\sin x}{x}$

C) $h(x)=x-[x]$ (Here, $[x]$ denotes the greatest integer function.)

D) $j(x)=\left|\sin\frac{1}{x}\right|$

E) $k(x)=\frac{3x^2-27}{3+x}$

ANSWER: A) $x = 3$; infinite
B) $x = 0$; removable
C) Integers; jump
D) $x = 0$; none of these
E) $x = -3$; removable

38. At each point of discontinuity, state whether the function is left or right continuous:

A) $f(x) = \begin{cases} \sin\left(\frac{\pi x}{2}\right) + 4 & |x| \le 2 \\ |x-2| & |x| > 2 \end{cases}$

B) $f(x) = \begin{cases} 1 & x \le 0 \\ \frac{\sin x}{x} & 0 < x \le \frac{\pi}{2} \\ \frac{2x}{\pi - x} & \frac{\pi}{2} < x < \pi \\ x - \pi & \pi \le x \end{cases}$

ANSWER: A) $x = 2$; left continuous
B) $x = \frac{\pi}{2}$; left continuous
$x = \pi$; right continuous

39. Determine real numbers *a, b,* and c that make the function continuous:

$$f(t) = \begin{cases} a & t < 0 \\ \frac{1}{4}t(t+8) & 0 \le t < b \\ t + 3 & b \le t < 4 \\ c & 4 \le t \end{cases}$$

ANSWER: $a = 0,\ b = 2,\ c = 7$

40. Find the points of discontinuity for each of these functions and state the type of discontinuity: removable, jump, infinite, or none of these.

A) $f(x) = \frac{|4+x|}{4+x}$

B) $g(x) = \frac{[x]}{x},\ x > 0$ (Here, [*x*] denotes the greatest integer function.)

C) $h(x) = \frac{1-x}{x^2+4x-5}$

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ANSWER: A) $x = -4$; jump
B) x = positive integer; jump
C) $x = 1$ removable; $x = -5$ infinite

41. Determine whether the function is left or right continuous at each of its points of discontinuity:

A) $f(x) = \begin{cases} \cos \pi x & |x| \le \frac{1}{2} \\ x - \frac{1}{2} & |x| > \frac{1}{2} \end{cases}$

B) $f(x) = x^2[x],\ x \ge 0$ (Here, $[x]$ denotes the greatest integer function.)

ANSWER: A) $x = -\frac{1}{2}$ right continuous
B) Right continuous at the positive integers

42. Determine real numbers $a,$ $b,$ and c that make the following function continuous:

$$f(t) = \begin{cases} t + a & t < 0 \\ t^2 + t + b + \frac{a}{2} & 0 \le t < 1 \\ t - b & 1 \le t < 2 \\ c & 2 \le t \end{cases}$$

ANSWER: $a = -\frac{2}{3},\ b = -\frac{1}{3},\ c = \frac{7}{3}$

43. Determine the points where the function is not continuous and state the type of the discontinuity: removable, jump, infinite, or none of these.

A) $f(x) = \dfrac{x^2 + 2x - 8}{|x - 2|}$

B) $g(x) = \dfrac{x}{[x]},\ x \ge 1$ (Here, $[x]$ denotes the greatest integer function.)

C) $h(x) = \dfrac{(x^3 - 3x + 2)\sin 2x}{x}$

D) $j(x) = \dfrac{4}{|x| - 3}$

ANSWER: A) $x = 2$, jump
B) $x = 2, 3, 4, \ldots$; jump
C) $x = 0$; removable
D) $x = 3$, $x = -3$; infinite

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44. At each point of discontinuity, state whether the function is left or right continuous.

A) $f(x)=\begin{cases} \sin\dfrac{1}{x} & x<0 \\ 1+x^2 & 0\le x<2 \\ (x+1)^2-4 & 2\le x<3 \\ 10 & 3\le x \end{cases}$

B) $f(x)=\begin{cases} |x-1| & x\le 2 \\ x^2-3 & 2<x\le 4 \\ \dfrac{1}{x-4} & 4<x<5 \\ 6 & 5\le x \end{cases}$

ANSWER: A) $x=0$; right continuous
$x=3$; right continuous
B) $x=4$; left continuous
$x=5$; right continuous

45. Determine real numbers a, b, and c that make the following function continuous:

$$f(x)=\begin{cases} a & x\le -1 \\ \dfrac{x}{[x]} & -1<x<0 \\ \dfrac{\sin\left(\frac{\pi x}{2}\right)+b+c}{x^2+1} & 0\le x<1 \\ b & 1\le x \end{cases}$$

(Here, $[x]$ denotes the greatest integer function.)

ANSWER: $a=1;\ b=\dfrac{1}{2};\ c=-\dfrac{1}{2}$

46. Determine the points where the function is not continuous and state the type of discontinuity: removable, jump, infinite, or none of these:

A) $f(x)=\dfrac{x^2+x-6}{x-2}$

B) $g(x)=\dfrac{1}{x-2}+\sin\dfrac{1}{x}$

C) $h(x)=[x]x$ (Here, $[x]$ denotes the greatest integer function.)

D) $j(x)=\dfrac{x^2+x-6}{x-3}$

ANSWER: A) $x = 2$; removable
B) $x = 2$; infinite
$x = 0$; none of these
C) Nonzero integers; jump
D) $x = 3$; infinite

47. At each point of discontinuity state whether the function is left continuous, right continuous, or neither

A) $$f(x) = \begin{cases} \dfrac{1}{x-2} & x < 1 \\ \cos \pi x & 1 \le x \le 2 \\ \dfrac{1+x}{(x-3)^2} & 2 < x \end{cases}$$

B) $$f(x) = \begin{cases} 0 & x < 0 \\ \cos \pi x & 0 \le x \le 1 \\ 2 \cos \pi x & 1 < x \le 2 \\ 2 & 2 < x \end{cases}$$

ANSWER: A) $x = 2$; left continuous
$x = 3$; none of these
B) $x = 0$; right continuous
$x = 1$; left continuous

48. Determine real numbers a, b, and c that make the function continuous:

$$f(x) = \begin{cases} a & t < 0 \\ x^2 + 1 & 0 \le t < b \\ 5x - c & b \le t < 7 \\ 42 & 7 \le t \end{cases}$$

ANSWER: $a = 1;\ b = 6;\ c = -7$

49. Consider the function

$$f(x) = \begin{cases} -1 & x \le 0 \\ 1 & x > 0 \end{cases}$$

The function $f(x) + g(x)$ is continuous for which of the following functions *g*?

a. $g(x) = 2$ if $x \ne 0,\ g(0) = 0$

b. $g(x) = 0$ if $x \ne 0,\ g(0) = 2$

c. $g(x) = 2$ if $x \le 0,\ g(x) = 0$ if $x > 0$

d. $g(x) = 2$ if $x < 0,\ g(x) = 0$ if $x \ge 0$

e. A and C both correct

ANSWER: c

50. Let $f(x)$ be a discontinuous function. Is it possible to find a continuous function $g(x)$ such that $f(x)+g(x)$ is continuous? Explain.

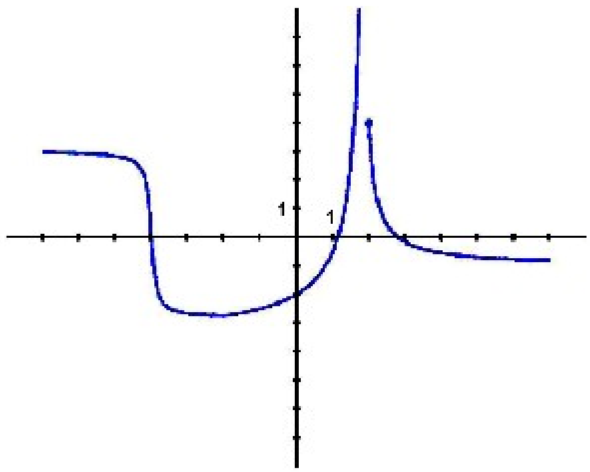
ANSWER: No. If $F(x) = f(x) + g(x)$ is continuous, then $f(x) = F(x) - g(x)$ is continuous by the continuity laws.

51. Sketch the graph of a function $f(x)$ that satisfies all of the following conditions:

$\lim_{x\to 2^-} f(x) = \infty$, $\lim_{x\to 2^+} f(x) = 4$, $\lim_{x\to 0} f(x) = -2$,

$\lim_{x\to \infty} f(x) = -1$, $\lim_{x\to -\infty} f(x) = 3$

ANSWER:



52. Evaluate each limit or state that it does not exist:

A) $\lim_{x\to 3} \dfrac{x^4-3x^3+x^2-9}{x-3}$

B) $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x^2+3x-1}-\sqrt{2x+1}}$

C) $\lim_{x\to 1} \dfrac{\sqrt{x^2-2x+1}}{x-1}$

ANSWER:

A) 33

B) $\dfrac{2\sqrt{3}}{3}$

C) Does not exist

53. Evaluate each limit or state that it does not exist:

A) $\lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{(1+\cos 2x)(2+\cos 2x)}$

B) $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{\sqrt{1+x^2}}{x^2} \right)$

C) $\lim_{\theta \to 0} \frac{1+\sin\theta-\cos\theta}{\sin\theta}$

ANSWER:

A) $\frac{1}{2}$

B) $-\frac{1}{2}$

C) 1

54. Evaluate the limits in terms of the constants involved:

A) $\lim_{x \to 0} \frac{\sqrt{(x+h)^2+1}-\sqrt{h^2+1}}{x}$

B) $\lim_{h \to a} \frac{\frac{1}{\sqrt{a}}-\frac{1}{\sqrt{h}}}{h-a}$, $a>0$

ANSWER:

A) $\frac{h}{\sqrt{1+h^2}}$

B) $\frac{1}{2a\sqrt{a}}$

55. Evaluate each limit or state that it does not exist:

A) $\lim_{x \to 2} \frac{x^4-x^2-12}{x-2}$

B) $\lim_{x \to 1} \frac{\sqrt{x^2+1}-\sqrt{x+1}}{x^2+x-2}$

C) $\lim_{x \to 2} \frac{\sqrt{x^2-4x+4}}{x-2}$

ANSWER:

A) 28

B) $\frac{\sqrt{2}}{12}$

C) Does not exist

56. Evaluate the limit:

$$\lim_{x\to 1} \frac{\sqrt{x+\sqrt{x}}-\sqrt{x+1}}{\sqrt{x}-1}$$

ANSWER: $\frac{\sqrt{2}}{4}$

57. Evaluate each limit or state that it does not exist:

A) $\lim_{x\to 2} \frac{x^3-2x^2+3x-6}{x-2}$

B) $\lim_{x\to -1} \frac{\sqrt{3x+4}+x}{x^2-x-2}$

C) $\lim_{x\to 1} \frac{|x^2+x-2|}{x-1}$

ANSWER:
A) 7
B) $-\frac{5}{6}$
C) Does not exist

58. Determine a real number c for which the limit exists and then compute the limit:

$$\lim_{x\to 0+}\left(\frac{1}{\sqrt{x+c^2}}-\frac{1}{\sqrt{x^2+x}}\right)$$

ANSWER: $c=0$, the limit is 0

59. Evaluate each limit or state that it does not exist:

A) $\lim_{x\to 2}\left(\frac{x+1}{x-2}-\frac{x-5}{x^2-5x+6}\right)$

B) $\lim_{x\to 5} \frac{\sqrt{x^2-6}-\sqrt{4x-1}}{x-5}$

C) $\lim_{x\to 8} \frac{\sqrt{x+1}-3}{x-8}$

ANSWER:
A) (-1)
B) $\frac{3\sqrt{19}}{19}$

C) $\frac{1}{6}$

60. Determine a real number a for which the limit exists and then compute the limit:
$\lim_{x \to 2} \frac{x^2 + ax + 6}{\sqrt{x^2 + 2x - 4} - \sqrt{x + 2}}$
ANSWER: $a = -5$; limit is $-\frac{4}{5}$

61. Let $f(x) = 2x + 3$. Compute $\lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h}$.
ANSWER: 2

62. Compute $\lim_{\theta \to 0} \left(\cot^2 \theta - \csc^2 \theta\right)$.
ANSWER: -1

63. Compute $\lim_{\theta \to \frac{7\pi}{6}} \frac{2 \sin^2 \theta - 5 \sin \theta - 3}{2 \sin \theta + 1}$.
ANSWER: $-\frac{7}{2}$

64. Evaluate the limits:
A) $\lim_{x \to 0} \frac{\sin^2 x}{\sin x^2}$
B) $\lim_{x \to 0} x \cos \frac{1}{x^3}$
C) $\lim_{x \to 0} |\sin x| \left(1 - \cos \frac{1}{x}\right)$
ANSWER:
A) 1
B) 0
C) 0

65. Show that $0 \le x - [x] < 1$ for all x. (Here, $[x]$ denotes the greatest integer function.)
Then use the above inequality and the Squeeze Theorem to evaluate $\lim_{x \to 0} x(x - [x])$.
ANSWER: 0

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66. Evaluate the limits in terms of the constants involved:

A) $\lim_{x \to h} \frac{\sin(x-h)}{x^2+(1-h)x-h}$

B) $\lim_{x \to a} \frac{\frac{1}{x^2}-\frac{1}{a^2}}{x-a}$

ANSWER:

A) $\frac{1}{h+1}$

B) $\left(-\frac{2}{a^3}\right)$

67. Evaluate the limits using the Squeeze Theorem, trigonometric identities, and trigonometric limits, as necessary:

A) $\lim_{x \to 0} \frac{\sin\frac{1}{x}\sin^2\frac{x}{2}}{x}$

B) $\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2}-x}$

C) $\lim_{x \to 0} \frac{\sin^3 x}{\sin\left(x^3\right)}$

ANSWER:

A) 0
B) 1
C) 1

68. Show that $0 \le x-[x] < 1$ for all x. (Here, $[x]$ denotes the greatest integer function.) Then use this inequality with the Squeeze Theorem to evaluate $\lim_{x \to \pi} (x-[x])\tan x$.

ANSWER: 0

69. Determine a real number c such that the following limit exists, and then evaluate the limit for this value:

$\lim_{x \to 0} \frac{3\sin\frac{x}{2}+(c-1)^2}{\sin x-\cos x+1}$

ANSWER:

$c = 1$; limit is $\frac{3}{2}$

70. Evaluate each limit or state that it does not exist:

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A) $\lim_{t\to 0} \frac{1-\cos t}{\sqrt{t}}$

B) $\lim_{x\to 0} \frac{1-\cos x}{\sin 2x \sin x}$

C) $\lim_{x\to 0} \frac{\sin 2x - \sin x}{x}$

ANSWER:

A) 0

B) $\frac{1}{4}$

C) 1

71. Evaluate the limits:

A) $\lim_{x\to 0} \frac{1-\cos^4 x}{x^2}$

B) $\lim_{x\to 0} \frac{\cos\left(x+\frac{\pi}{4}\right)-\sin\left(x+\frac{\pi}{4}\right)}{x}$

ANSWER:

A) 2

B) $-\sqrt{2}$

72. Evaluate the limits:

A) $\lim_{x\to 0} \frac{\sin 5x}{x^2-x}$

Hint: Factor the denominator.

B) $\lim_{x\to 0} \frac{1-\cos^2 x}{2x^2-x}$

Hint: Factor the two expressions.

C) $\lim_{x\to 0} \frac{\sin 3x \sin 8x}{x^3+x^2}$

ANSWER:

A) -5

B) 0

C) 24

73. Use the Squeeze Theorem to evaluate the limit

$\lim_{x\to \pi} (1+\cos x) \sin \frac{1}{x}$

ANSWER: 0

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74. If $3x^2-4 \le f(x) \le x$ on the interval [0,4], then $\lim_{x \to 1} f(x)$ must exist.

a. True

b. False

ANSWER: b

75. Calculate the limits:

A) $\lim_{x \to \infty} \frac{2x^5-x^4+1}{8x^5+x^3+x-2}$

B) $\lim_{x \to -\infty} \frac{3x^2+x-1}{4x-7}$

C) $\lim_{x \to \infty} \left(\frac{6x^3}{2x^2+1} - 3x \right)$

ANSWER:

A) $\frac{1}{4}$

B) $-\infty$

C) 0

76. Calculate the limits:

A) $\lim_{x \to -\infty} \frac{\sqrt[3]{2x^3-x+1}}{\sqrt{x^2+x-2}}$

B) $\lim_{x \to \infty} \frac{(4x+1)^{15}(3x-1)^{10}}{(9x+7)^5(4x+11)^{20}}$

C) $\lim_{x \to -\infty} \frac{\sqrt{|x^2-5|}}{x}$

ANSWER:

A) $-\sqrt[3]{2}$

B) $\frac{1}{1024}$

C) -1

77. Calculate the following limits:

A) $\lim_{x \to -\infty} \frac{x^2+1}{\sqrt{x^4-2}}$

B) $\lim_{x \to -\infty} \frac{2x-1}{\sqrt[3]{x^3+1}}$

C) $\lim_{x \to \infty} \left(\sqrt{x^2 - x} - \sqrt{x^2 + 5x}\right)$

ANSWER:

A) 1
B) 2
C) -3

78. Compute the following limits:

A) $\lim_{x \to \infty} \dfrac{2x^2 - 6x + 1}{x^2 - 3}$

B) $\lim_{x \to -\infty} \left(-x^5 + 2x^4 - x^2 + 1\right)$

C) $\lim_{x \to -\infty} \dfrac{x^7 - 6x^3 + 1}{2x^7 + x^2 - 2}$

ANSWER:

A) 2
B) ∞
C) $\dfrac{1}{2}$

79. Compute the following limits:

A) $\lim_{x \to \infty} \dfrac{3(x+7)^3 - (x-7)^3}{2(x+2)^3 - (x-2)^3}$

B) $\lim_{x \to -\infty} \dfrac{x - 1}{2x^2 + 1}$

C) $\lim_{x \to -\infty} \dfrac{2x^2 - 6x + 1}{x - 3}$

ANSWER:

A) 2
B) 0
C) $-\infty$

80. Compute the following limits:

A) $\lim_{x \to \infty} x\left(\sqrt{4x^2 - 1} - 2x\right)$

Hint: Multiply and divide by the conjugate expression.

B) $\lim_{x \to -\infty} \dfrac{2x + 7}{\sqrt{x^2 - 1}}$

Hint: For $x < 0, \sqrt{x^2} = -x$.

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C) $\lim_{x \to \infty} \dfrac{x^{\frac{5}{3}} - 3x^{\frac{2}{3}}}{x^{\frac{18}{5}} + x}$

ANSWER:

A) $-\dfrac{1}{4}$
B) -2
C) 0

81. Compute the following limits:

A) $\lim_{x \to \infty} \left(\dfrac{1}{x-1} - \dfrac{2}{x^2-1} \right)$

B) $\lim_{x \to -\infty} \dfrac{x^3 + x^2 - 2x + 1}{1 - 2x^3}$

C) $\lim_{x \to -\infty} \dfrac{x^4 + 2x - 1}{x^3 + x}$

ANSWER:

A) 0
B) $-\dfrac{1}{2}$
C) $-\infty$

82. Compute the following limits:

A) $\lim_{x \to -\infty} \dfrac{1 - \sqrt{3 + x^2}}{1 + \sqrt{4x^2 + 1}}$

B) $\lim_{x \to \infty} \dfrac{x + 3 - \sqrt{x^2 + 2}}{\sqrt{x^2 + 1} - 5}$

(C) $\lim_{x \to \infty} \left(\sqrt{x^2 - 3x + 7} - x \right)$

ANSWER:

A) $-\dfrac{1}{2}$
B) 0
C) $-\dfrac{3}{2}$

83. The Intermediate Value Theorem guarantees that the equation $x \cos x - \sin x = 0$ has a solution in which of the following intervals?

a. $(2\pi, 3\pi)$

b. $\left(\frac{\pi}{2}, \pi\right)$

c. $\left(\frac{3\pi}{2}, 2\pi\right)$

d. $\left(\frac{\pi}{4}, \pi\right)$

e. $(\pi, 3\pi)$

ANSWER: a

84. The polynomial $P(x) = x^3 - x - 5$ must have a root in which of the following intervals?

a. $(3, 4)$

b. $(1, 2)$

c. $(0, 1)$

d. $\left(\frac{1}{2}, 1\right)$

e. $(-1, 1)$

ANSWER: b

85. Which of the following functions is a counterexample for the converse of the Intermediate Value Theorem, which states: If $f(x)$ assumes all the values between $f(a)$ and $f(b)$ in the interval $[a, b]$, then f is continuous on $[a, b]$.

a. $f(x) = x - 1$ on $[0, 2]$

b. $f(x) = \frac{1}{x-1}$ on $[0, 2]$

c. $f(x) = \frac{\sin x}{x}$ on $(0, 2), f(0) = 1$ $f(x) = \begin{cases} \frac{\sin x}{x}, 0 < x \le 2 \\ 1, x = 0 \end{cases}$ on [0,2]

d. $f(x) = [x]$ on $[0, 2]$ (Here, $[x]$ denotes the greatest integer function.)

e. $f(x) = \frac{1}{(x-1)^2}$ on $[-3, 2]$

ANSWER: e

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86. Which of the following functions has a zero in the interval $[-1, 3]$?

a. $f(x) = \dfrac{x}{x-4}$

b. $f(x) = x^2 - 3x + 3$

c. $f(x) = \dfrac{x^2}{x-2}$

d. $f(x) = \cos\dfrac{x}{\pi}$

e. Both A and C

ANSWER: e

87. The Intermediate Value Theorem guarantees that the equation $\tan x = x$ has a solution in which of the following intervals?

a. $\left[-\dfrac{\pi}{8}, \dfrac{3\pi}{8}\right]$

b. $\left[\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$

c. $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

d. $\left[\dfrac{\pi}{4}, \pi\right]$

e. Both A and C

ANSWER: e

88. Which of the following functions is a counterexample for the converse of the Intermediate Value Theorem, which states: If $f(x)$ assumes all the values between $f(a)$ and $f(b)$ in the interval $[a, b]$, then f is continuous on $[a, b]$.

a. $f(x) = x^2$ for $x \in (1, 3)$, $f(1) = 9$, $f(3) = 1$ on [1,3]

b. $f(x) = \dfrac{1}{x-2}$ on $[1, 3]$

c. $f(x) = \dfrac{1 - \cos x}{x}$ on $\left(0, \dfrac{\pi}{2}\right]$, $f(0) = 0$ on $\left[0, \dfrac{\pi}{2}\right]$

d. $f(x) = [x]$ on $[1, 3]$ (Here, $[x]$ denotes the greatest integer function.)

e. Both A and C

ANSWER: a

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89. Which of the following functions is a counterexample for the converse of the Intermediate Value Theorem: If f assumes all the values between $f(a)$ and $f(b)$ in the interval $[a, b]$, then f is continuous on $[a, b]$.

a. $f(x) = \dfrac{1}{x-1}$ if $1 < x \leq 3,\ f(1) = 2$ on [1,3]

b. $f(x) = [x]$ on $1 < x \leq 3$ (Here, [*x*] denotes the greatest integer function.)

c. $f(x) = \dfrac{1}{x-4}$ on $1 < x \leq 3$

d. $f(x) = x^2$ for $1 < x \leq 3$ and $x \neq 2,\ f(2) = 1$

e. Both A and D

ANSWER: a

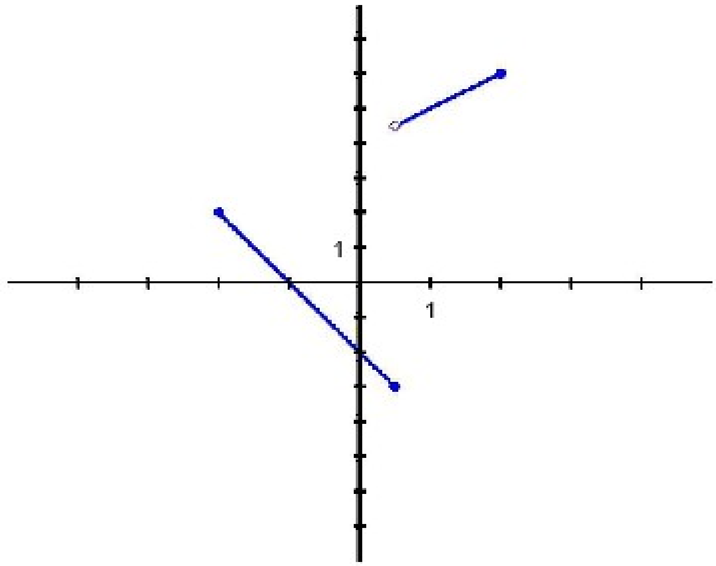
90. Assume $g(x)$ is continuous on $[-3, 9]$, $g(-3) = 14$, and $g(9) = 72$. Determine whether each of the following statements is always true, never true, or sometimes true.

A) $g(c) = 0$: no solution with $c \in [-3, 9]$
B) $g(c) = 60$: no solution with $c \in [-2, 9]$
C) $g(c) = 21$: no solution with $c \in [-3, 9]$
D) $g(c) = -1{,}000{,}000$: exactly one solution with $c \in [-2, 9]$
E) $g(c) = 49.5$: a solution with $c \in [-3, 9]$

ANSWER: A) Sometimes true
B) Sometimes true
C) Never true
D) Sometimes true
E) Always true

91. Draw the graph of a function $g(x)$ on $[-2, 2]$ such that the graph does not satisfy the conclusion of the Intermediate Value Theorem.

ANSWER: Answers may vary. A sample answer is:



92. Which of the following properties can be used to prove that $f(x) = \cos x$ is continuous for all x?

a. $|\cos x| \le 1$ for all x

b. $|\cos x - \cos y| \le |x - y|$ for all x and y

c. $\cos x - \cos y \le x - y$ for all x and y

d. The limit $\lim_{x \to 0} \frac{1 - \cos x}{x}$ exists

e. $|\cos x - \cos y| \ge |x - y|$ for all x and y

ANSWER: b

93. Which of the following statements imply that $\frac{1}{x}$ is not continuous at $x = 0$?

a. $\frac{1}{x}$ has opposite signs on the two sides of $x = 0.$

b. $|\frac{1}{x}| < 0.01$ implies that $x > 100.$

c. For any $\varepsilon > 0$, $|\frac{1}{x}| < \varepsilon$ implies that $|x| > \frac{1}{\varepsilon}.$

d. If $x < \varepsilon$, then $|\frac{1}{x}| < \frac{1}{\varepsilon}$.

e. A and C are correct.

ANSWER: c

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94. To show that L is not the limit of $f(x)$ as $x \to x_0$, we should show that:

a. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - x_0| > \delta$ then $|f(x) - L| < \varepsilon$.

b. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - x_0| > \delta$ then $|f(x) - L| > \varepsilon$.

c. There exists $\varepsilon > 0$, such that for any $\delta > 0$ the inequalities $0 < |x - x_0| < \delta$ and $|f(x) - L| \geq \varepsilon$ have a solution x.

d. There exist $\varepsilon > 0$ and $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $|f(x) - L| > \varepsilon$.

e. A and C are both correct.

ANSWER: c

95. Suppose there exists a value of $\varepsilon > 0$ so that for any value of $\delta > 0$, we can find a value of x satisfying $0 < |x - x_0| < \delta$ and $|f(x) - L| > \varepsilon$. We may conclude that:

a. L is the limit of f as $x \to x_0$.

b. L is not the limit of f as $x \to x_0$.

c. The limit of f as $x \to x_0$ does not exist.

d. The limit of f as $x \to x_0$ exists but is not equal to L.

e. None of the above.

ANSWER: b

96. To show that L is not the limit of $f(x)$ as $x \to x_0$, we should show that:

a. There exists $\varepsilon > 0$ such that for any $\delta > 0$ there exists a solution to the inequalities $|x - x_0| < \delta$ and $|f(x) - L| > \varepsilon$.

b. There exists $\varepsilon > 0$ such that for any $\delta > 0$ there exists a solution to the inequalities $|x - x_0| > \delta$ and $|f(x) - L| < \varepsilon$.

c. There exists $\delta > 0$ such that for any $\varepsilon > 0$, if $|f(x) - L| < \varepsilon$, then $|x - x_0| < \delta$.

d. For any $\varepsilon > 0$ and $\delta > 0$, if $|x - x_0| < \delta$, then $|f(x) - L| > \varepsilon$.

e. A and D are both correct.

ANSWER: a

