Chapter 1: Systems of Linear Equations and Matrices

Multiple Choice Questions

1. Which of the following equations is linear?

(A) $2x_1^2 + 3x_2^3 + 4x_3^4 = 5$

(B) $\sqrt{3}x_1 - \sqrt{2}x_2 + x_3 = 5$

(C) $\sqrt{5}x_1 + 5\sqrt{x_2} - x_3 = 1$

(D) $2^2x_1 + \cos(x_2) + 4x_3 = 7$

2. Which system corresponds to the following augmented matrix?

$$\begin{bmatrix} 1 & 11 & 6 & 3 \\ 9 & 4 & 0 & -2 \end{bmatrix}$$

(A) $\begin{aligned} x_1 + 11x_2 &= -3 \\ 9x_1 + 4x_2 &= -2 \end{aligned}$

(B) $\begin{aligned} x_1 + 11x_2 + 6x_3 &= 3 \\ 9x_1 + 4x_2 \qquad &= -2 \end{aligned}$

(C) $\begin{aligned} x_1 + 11x_2 + 6x_3 + 3x_4 &= 0 \\ 9x_1 + 4x_2 \qquad - 2x_4 &= 0 \end{aligned}$

(D) $\begin{aligned} x_1 + 9x_2 &= 0 \\ 11x_1 + 4x_2 &= 0 \\ 6x_1 \qquad &= 0 \\ 3x_1 - 2x_2 &= 0 \end{aligned}$

3. Which of the following statements best describes the following augmented matrix?

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ -1 & 1 & -2 & 3 \\ 1 & -4 & -2 & 1 \end{bmatrix}$$

(A) A is consistent with a unique solution.

(B) A is consistent with infinitely many solutions.

(C) A is inconsistent.

(D) none of the above.

4. Which of the following matrices is in *reduced* row echelon form?

(A) $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -7 & 5 \\ 0 & 0 & 1 & 14 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 & 11 & -3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

5. If the matrix A is 4×2, B is 3×4, C is 2×4, D is 4×3, and E is 2×5, which of the following expressions is *not* defined?

(A) $A^T D + CB^T$ (B) $(B + D^T)A$ (C) $CA + CB^T$ (D) $DBAE$

6. What is the second row of the product AB?

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 5 & 4 & 8 \\ 9 & 7 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 1 & 7 \\ 6 & 3 & 2 \\ 2 & 9 & 7 \end{bmatrix}$$

(A) $\begin{bmatrix} 18 & 33 & 25 \end{bmatrix}$ (B) $\begin{bmatrix} 64 & 48 & 91 \end{bmatrix}$ (C) $\begin{bmatrix} 50 & 89 & 99 \end{bmatrix}$ (D) $\begin{bmatrix} 48 & 89 & 33 \end{bmatrix}$

7. Which of the following is the determinant of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

(A) $ad - bc$ (B) $bc - ad$ (C) $\frac{1}{bc-ad}$ (D) $\frac{1}{ad-bc}$

8. Which of the following matrices is *not* invertible?

(A) $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$ (B) $\begin{bmatrix} 7 & 7 \\ 2 & 3 \end{bmatrix}$ (C) $\begin{bmatrix} 9 & 0 \\ 4 & 4 \end{bmatrix}$ (D) $\begin{bmatrix} 9 & 3 \\ 6 & 5 \end{bmatrix}$

9. Which of the following matrices is *not* an elementary matrix?

(A) $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ (C) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. For which elementary matrix E will the equation $EA = B$ hold?

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 2 & 10 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 2 & -3 \end{bmatrix}$$

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

11. Which matrix will be used as the inverted coefficient matrix when solving the following system?

$$\begin{aligned} 3x_1 + x_2 &= 4 \\ 5x_1 + 2x_2 &= 7 \end{aligned}$$

(A) $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ (B) $\begin{bmatrix} -2 & 1 \\ 5 & -3 \end{bmatrix}$ (C) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (D) $\begin{bmatrix} -2 & -1 \\ -5 & -3 \end{bmatrix}$

12. What value of b makes the following system consistent?

$$\begin{aligned} 4x_1 + 2x_2 &= b \\ 2x_1 + x_2 &= 0 \end{aligned}$$

(A) $b = -1$ (B) $b = 0$ (C) $b = 1$ (D) $b = 2$

13. If A is a 3×3 diagonal matrix, which of the following matrices is *not* a possible value of A^k for some integer k?

(A) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 16 & 0 \\ 4 & 0 & 25 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

14. The matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is:

(A) upper triangular.
(B) lower triangular.
(C) both (A) and (B).
(D) neither (A) nor (B).

15. If A is a 4×5 matrix, find the domain and codomain of the transformation $T_A(\mathbf{x}) = A\mathbf{x}$.

(A) Not enough information

(B) Domain: R^4, Codomain: R^5

(C) Domain: R^5, Codomain: R^5

(D) Domain: R^5, Codomain: R^4

16. Which of the following is a matrix transformation?

(A) $T(x, y, z) = (yx^2, yz^2)$

(B) $T(x, y, z, w) = (xy, yz, zw, wx)$

(C) $T(x, y, z) = (x + 1, x + 2, x + z, y + z)$

(D) $T(x, y) = (4x, 5x, -x, 0)$

17. Which matrix represents reflection about the xy-plane?

(A) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

18. Use matrix multiplication to find the image of the vector $(2, 1)$ when it is rotated counterclockwise about the origin through an angle $\theta = 45°$.

(A) $\left(\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ (B) $\left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ (C) $\left(-\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ (D) $\left(-\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

19. Which of the following pairs of operators $T_1, T_2 : R^2 \to R^2$ commute? (That is, for which pair is it true that $T_1 \circ T_2 = T_2 \circ T_1$?)

(A) T_1 is the reflection about the x-axis.
T_2 is the reflection about line $y = x$.

(B) T_1 is the orthogonal projection onto the x-axis.
T_2 is the reflection about line $y = x$.

(C) T_1 is the counterclockwise rotation about the origin through an angle of π.
T_2 is the projection onto the y-axis.

(D) T_1 is the reflection about the x-axis.
T_2 is the counterclockwise rotation about the origin through an angle of $\pi/2$.

Free Response Questions

1. Find the relationship between a and b such that the following system has infinitely many solutions.

$$-x + 2y = a$$
$$-3x + 6y = b$$

2. Solve the following system and use parametric equations to describe the solution set.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 11 \\ 2x_1 - x_2 + x_3 &= 2 \\ 3x_1 + x_2 + 4x_3 &= 13 \end{aligned}$$

3. Determine whether the following system has no solution, exactly one solution, or infinitely many solutions.

$$\begin{aligned} 2x_1 + 2x_2 &= 2 \\ x_1 + x_2 &= 4 \end{aligned}$$

4. Find the value of k that makes the system $\begin{bmatrix} 15 & -3 & 6 \\ -10 & k & 9 \end{bmatrix}$ inconsistent.

5. Solve the following system using Gaussian elimination.

$$\begin{aligned} x_1 - x_2 - 5x_3 &= -1 \\ -2x_1 + 2x_2 + 11x_3 &= 1 \\ 3x_1 - x_2 + x_3 &= 3 \end{aligned}$$

6. Solve the following system for x, y, and z.

$$\begin{aligned} \tfrac{1}{x} - \tfrac{1}{y} - \tfrac{1}{z} &= 0 \\ \tfrac{2}{x} + \tfrac{1}{y} + \tfrac{1}{z} &= 3 \\ \tfrac{3}{x} \qquad - \tfrac{1}{z} &= 0 \end{aligned}$$

7. The curve $y = ax^3 + bx^2 + x + c$ passes through the points $(0,0), (1,1)$, and $(-1,-2)$. Find and solve a system of linear equations to determine the values of a, b, and c.

8. Solve the following system for x and y.

$$\begin{aligned} x^2 + y^2 &= 6 \\ x^2 - y^2 &= 2 \end{aligned}$$

9. Given $C = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, find CC^T.

10. Express the following matrix equation as a system of linear equations.

$$\begin{bmatrix} -1 & 7 & 0 \\ 0 & 4 & 3 \\ 6 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

11. Find the 3×3 matrix $A = [a_{ij}]$ whose entries satisfy the condition $a_{ij} = i^2 - j$.

12. Let A and B be $n \times n$ matrices. Prove that $\operatorname{tr}(c \cdot A - B) = c \cdot \operatorname{tr}(A) - \operatorname{tr}(B)$.

13. What is the inverse of $\begin{bmatrix} 4 & 0 \\ 9 & 2 \end{bmatrix}$?

14. Given the polynomial $p(x) = x^2 - 3x + 1$ and the matrix $A = \begin{bmatrix} 4 & 4 \\ 6 & 1 \end{bmatrix}$, compute $p(A)$.

15. Let $A, B, C,$ and D be $n \times n$ invertible matrices. Solve for A given that the following equation holds.

$$C^2 D A^{-1} C B^{-1} = B C B^{-1}$$

16. Prove that for any $m \times n$ matrices A and B, $(A - B)^T = A^T - B^T$.

17. Use the inversion algorithm to find the inverse of the following matrix.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

18. Which elementary row operation will transform the following matrix into the identity matrix?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -9 & 0 & 1 \end{bmatrix}$$

19. Find the 3×3 elementary matrix that adds c times row 3 to row 1.

20. Find the elementary matrix E that satisfies

$$E \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 2 & 10 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 2 & -3 \end{bmatrix}$$

21. Solve the following system by inverting the coefficient matrix.

$$\begin{aligned} 7x + 2y &= 1 \\ 3x + y &= 5 \end{aligned}$$

22. Solve the following matrix equation for X.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} X = \begin{bmatrix} 2 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

23. Given that $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, solve the system $A^2\mathbf{x} = \mathbf{b}$.

24. Find a nonzero solution to the following equation.

$$\begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix} \mathbf{x} = 3\,\mathbf{x}$$

25. Find the values of a, b, and c that make the following matrix symmetric.

$$\begin{bmatrix} 3 & a & 2-b \\ 4 & 0 & a+b \\ 2 & c & 7 \end{bmatrix}$$

26. Let $A = \begin{bmatrix} 3 & 4 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -7 & 6 \\ -4 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, and $AB = [c_{ij}]$.

Find the diagonal entries c_{11}, c_{22}, and c_{33}.

27. Let the entries of a matrix $A = [a_{ij}]$ be defined as $a_{ij} = 2i^2 - i + j + g(j)$, where g is a function of j. If A is a symmetric matrix, what is $g(j)$?

28. Prove that for any square matrix A, the matrix $B = (A + A^T)$ is symmetric.

29. Find the domain and codomain of the transformation defined by

$$\begin{bmatrix} 5 & 7 & 6 & 0 \\ 1 & 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

30. Find the standard matrix for the operator $T : R^2 \to R^2$ defined by

$$\begin{aligned} 3x_1 + x_2 &= w_1 \\ 4x_2 &= w_2 \end{aligned}$$

31. Find the standard matrix for the transformation T defined by the formula

$$T(x_1, x_2, x_3) = (x_1, -x_3, x_2 - x_1, 3x_2 + x_3)$$

32. Find the standard matrix A for the linear transformation $T : R^2 \to R^2$ for which

$$\mathrm{T}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \left(\begin{bmatrix} 3 \\ -4 \end{bmatrix}\right), \mathrm{T}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right)$$

33. Prove that the composition of two rotation operators about the origin of R^2 is another rotation about the origin.

34. Prove that if $T_A : R^3 \to R^3$ and $T_A(\mathbf{x}) = \mathbf{0}$ for every vector $\mathbf{x}$ in R^3, then A is the 3×3 zero matrix.

35. Write a balanced equation for the following chemical reaction.

$$\mathrm{C_3H_8 + O_2 \to H_2O + CO_2}$$

36. Find the quadratic polynomial whose graph passes through the points $(0,\ 3)$, $(-1,\ 8)$, and $(1,\ 0)$.

37. Use matrix inversion to find the production vector $\mathbf{x}$ that meets the demand $\mathbf{d}$ for the consumption matrix C.

$$C = \begin{bmatrix} 0.1 & 0.3 & 0.2 \\ 0.5 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.3 \end{bmatrix};\ \mathbf{d} = \begin{bmatrix} 18 \\ 40 \\ 26 \end{bmatrix}$$

Answers

Multiple Choice Answers

1. (B)

2. (B)

3. (C)

4. (D)

5. (C)

6. (C)

7. (A)

8. (A)

9. (B)

10. (C)

11. (A)

12. (B)

13. (B)

14. (C)

15. (D)

16. (D)

17. (C)

18. (A)

19. (C)

Free Response Answers

1. $3a = b$

2. $x_1 = -t + 3\ ,\ x_2 = -t + 4\ ,\ x_3 = t$

3. no solution

4. $k = 2$

5. $x_1 = 5,\ x_2 = 11,\ x_3 = -1$

6. $x = 1,\ y = -\frac{1}{2},\ z = \frac{1}{3}$

7. System:
$$\begin{aligned} c &= 0 \\ a + b + c &= 0 \\ -a + b + c &= -1 \end{aligned}$$

Solution: $a = \frac{1}{2}, b = -\frac{1}{2}, c = 0$

8. $x = \pm 2,\ y = \pm\sqrt{2}$

9. $CC^T = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$

10.
$$\begin{aligned} -x + 7y \phantom{{}+3z} &= 0 \\ 4y + 3z &= 0 \\ 6x \phantom{{}+4y} - 2z &= 0 \end{aligned}$$

11. $A = \begin{bmatrix} 0 & -1 & -2 \\ 3 & 2 & 1 \\ 8 & 7 & 6 \end{bmatrix}$

13. $\begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{9}{8} & \frac{1}{2} \end{bmatrix}$

14. $\begin{bmatrix} 29 & 8 \\ 12 & 23 \end{bmatrix}$

15. $A = B^{-1}C^2D$

17. $\begin{bmatrix} 1 & -1 & \frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

18. Add 9 times row 2 to row 4

19. $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

20. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

21. $x = -9$, $y = 32$

22. $\begin{bmatrix} -33 & -43 & -67 & 23 \\ 28 & 36 & 56 & -19 \\ -7 & -9 & -14 & 5 \end{bmatrix}$

23. $\mathbf{x} = \begin{bmatrix} 3 \\ 9 \\ 4 \end{bmatrix}$

24. Any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $2x_1 = 3x_2$. Possible solution: $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

25. $a = 4,\ b = 0,\ c = 4$

26. $c_{11} = -10$, $c_{22} = 0$, and $c_{33} = 4$

27. $g(j) = 2j^2 - 2j$

29. Domain: R^4, Codomain: R^2

30. $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$

32. $\begin{bmatrix} 8 & 13 \\ -2 & 0 \end{bmatrix}$

35. $C_3H_8 + 5O_2 \to 4H_2O + 3CO_2$

36. $3 - 4x + x^2$

37. $\mathbf{x} \approx \begin{bmatrix} 91.85 \\ 125.50 \\ 135.10 \end{bmatrix}$