REVISION TEST 2 (Page 78)

A.P.'s, G.P.'s, binomial and Maclaurin's theorems

This assignment covers the material contained in chapters 6 to 8.

1. Determine the 20^{th} term of the series 15.6, 15, 14.4, 13.8, ...

	Marks
The 20^{th} term is given by: $a + (n - 1)d$	1
i.e. $15.6 + (20 - 1)(- 0.6)$	1
$= 15.6 - 19(0.6) = 15.6 - 11.4 = \mathbf{4.2}$	1
Total:	**3**

2. The sum of 13 terms of an arithmetic progression is 286 and the common difference is 3.

Determine the first term of the series.

	Marks
$S_n = \frac{n}{2}[2a + (n - 1)d]$	1
i.e. $286 = \frac{13}{2}[2a + (13 - 1)3]$	1
$286 = \frac{13}{2}[2a + 36]$	
$\frac{286 \times 2}{13} = 2a + 36$ i.e. $44 - 36 = 2a$	2
from which, first term, $\mathbf{a} = \frac{44-36}{2} = \mathbf{4}$	
Total:	**4**

3. An engineer earns £21,000 per annum and receives annual increments of £600. Determine the salary in the 9th year and calculate the total earnings in the first 11 years.

	Marks
If first term a = £21,000 and the n'th term n = 9	
then salary in 9th year = a + (n - 1)d	
$= 21000 + (9 - 1)(600)$	
$= 21000 + 8(600) =$ **£25,800**	2
Total earnings in first 11 years, $S_{11} = \frac{n}{2}[2a + (n - 1)d]$	
$= \frac{11}{2}[2(21000) + (11 - 1)600]$	
$= \frac{11}{2}[42000 + 6000]$	3
= **£264,000**	
Total:	**5**

4. Determine the 11th term of the series 1.5, 3, 6, 12, ...

	Marks
The 11th term is given by : ar^{n-1} where a = 1.5 and common ratio r = 2	
i.e. 11th term = $(1.5)(2)^{11-1}$ = **1536**	2
Total:	**2**

5. Find the sum of the first eight terms of the series 1, 2.5, 6.25, ... , correct to 1 decimal place.

	Marks
In the series $1, 2\frac{1}{2}, 6\frac{1}{4}$, .., common ratio r = 2.5	1

and the sum of the first eight terms, $S_8 = \frac{a(r^n - 1)}{(r-1)} = \frac{1(2.5^8 - 1)}{2.5 - 1}$	3
$= \frac{1524.878..}{1.5}$ = **1016.6**	**4**
Total:	

6. Determine the sum to infinity of the series 5, 1, $\frac{1}{5}$, ...

	Marks
$S_\infty = \frac{a}{1-r}$ where a = 5 and r = $\frac{1}{5}$	3
Thus the sum to infinity, $S_\infty = \frac{5}{1-\frac{1}{5}} = \frac{5}{\frac{4}{5}} = \frac{25}{4} = \mathbf{6\frac{1}{4}}$	**3**
Total:	

7. A machine is to have seven speeds ranging from 25 rev/min to 500 rev/min. If the speeds form a geometric progression, determine their value, each correct to the nearest whole number.

	Marks
The G.P. of n terms is given by: a, ar, ar^2, .. ar^{n-1}	
The first term a = 25 rev/min	
The seventh term is given by ar^{7-1} which is 500 rev/min	
i.e. $ar^6 = 500$ from which, $r^6 = \frac{500}{a} = \frac{500}{25} = 20$	
thus the common ratio r = $\sqrt[6]{20}$ = 1.64755	3
The first term is 25 rev/min	
The second term ar = (25)(1.64755) = 41.19	
The third term $ar^2 = (25)(1.64755)^2 = 67.86$	

The fourth term $ar^3 = (25)(1.64755)^3 = 111.80$ The fifth term $ar^4 = (25)(1.64755)^4 = 184.20$ The sixth term $ar^5 = (25)(1.64755)^5 = 303.48$ Hence, correct to the nearest whole number the speeds of the machine are: **25, 41, 68, 112, 184, 303 and 500 rev/min** **Total:**	4 1 **8**

8. Use the binomial series to expand $(2a-3b)^6$

	Marks
$(2a-3b)^6 = (2a)^6 + 6(2a)^5(-3b) + \frac{(6)(5)}{2!}(2a)^4(-3b)^2 + \frac{(6)(5)(4)}{3!}(2a)^3(-3b)^3$ $+\frac{(6)(5)(4)(3)}{4!}(2a)^2(-3b)^4 + \frac{(6)(5)(4)(3)(2)}{5!}(2a)(-3b)^5 + (-3b)^6$ $= \mathbf{64a^6 - 576a^5b + 2160a^4b^2 - 4320a^3b^3 + 4860a^2b^4 - 2916ab^5 + 729b^6}$ **Total:**	4 3 **7**

9. Determine the middle term of $\left(3x - \frac{1}{3y}\right)^{18}$

	Marks
The middle term is the 10th i.e. $\frac{(18)(17)(16)(15)(14)(13)(12)(11)(10)}{9!}(3x)^9\left(-\frac{1}{3y}\right)^9 = \mathbf{-48620\frac{x^9}{y^9}}$ or $\mathbf{-48620\left(\frac{x}{y}\right)^9}$ **Total:**	1 5 **6**

10. Expand the following in ascending powers of t as far as the term in t^3:

(a) $\frac{1}{1+t}$ (b) $\frac{1}{\sqrt{1-3t}}$

For each case, state the limits for which the expansion is valid.

	Marks
(a) $\frac{1}{1+t} = (1+t)^{-1} = 1 + (-1)t + \frac{(-1)(-2)}{2!}t^2 + \frac{(-1)(-2)(-3)}{3!}t^3 \ldots$	
$= \mathbf{1 - t + t^2 - t^3 + \ldots}$	4
The expansion is valid when $\mathbf{\lvert t \rvert < 1}$ or $\mathbf{-1 < t < 1}$	1
(b) $\frac{1}{\sqrt{1-3t}} = (1-3t)^{-1/2} = 1 + \left(-\frac{1}{2}\right)(-3t) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-3t)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-3t)^3 \ldots$	5
$= \mathbf{1 + \frac{3}{2}t + \frac{27}{8}t^2 + \frac{135}{16}t^3 + \ldots}$	1
The expansion is valid when $\mathbf{\lvert 3t \rvert < 1}$ i.e. $\mathbf{\lvert t \rvert < \frac{1}{3}}$ or $\mathbf{-\frac{1}{3} < t < \frac{1}{3}}$	
Total:	**11**

11. When x is very small show that: $\frac{1}{(1+x)^2\sqrt{1-x}} \approx 1 - \frac{3}{2}x$

	Marks
$\frac{1}{(1+x)^2\sqrt{1-x}} = (1+x)^{-2}(1-x)^{-\frac{1}{2}} = [1 + (-2)x + \ldots]\left(1 - \left(-\frac{1}{2}\right)x + \ldots\right)$	
$\approx 1 - \left(-\frac{1}{2}\right)x + (-2)x + \ldots \approx 1 + \frac{1}{2}x - 2x + \ldots = \mathbf{1 - \frac{3}{2}x}$	5
when powers of 2 and above are neglected.	
Total:	**5**

12. The modulus of rigidity G is given by $G = \frac{R^4\theta}{L}$ where R is the radius, θ the angle of twist and L the length. Find the approximate percentage error in G when R is measured 1.5% too large, θ is measured 3% too small and L is measured 1% too small.

	Marks
The new values of R, θ and L are: (1 + 0.015)R, (1 – 0.03)θ and (1 – 0.01)L	
New modulus of rigidity = $\dfrac{[(1+0.015)R]^4[(1-0.03)\theta]}{[(1-0.01)L]}$	3
$= [(1+0.015)R]^4[(1-0.03)\theta][(1-0.01)L]^{-1}$	
$= (1+0.015)^4 R^4 (1-0.03)\theta(1-0.01)^{-1} L^{-1}$	
$= (1+0.015)^4(1-0.03)(1-0.01)^{-1} R^4 \theta L^{-1}$	
$= [1 + 4(0.015)][1 - 0.03][1 - (-1)(0.01)] \dfrac{R^4\theta}{L}$ neglecting products of small terms	
= [1 + 0.06 – 0.03 + 0.01] G = (1 + 0.04) G	5
i.e. **G is increased by 4%**	**8**
Total:	

13. Use Maclaurin's series to determine a power series for $e^{2x}\cos 3x$ as far as the term in x^2

	Marks
Let f(x) = $e^{2x}\cos 3x$ **f(0)** = $e^0 \cos 0$ = **1**	1
$f'(x) = (e^{2x})(-3\sin 3x) + (\cos 3x)(2e^{2x})$	
$= -3e^{2x}\sin 3x + 2e^{2x}\cos 3x$ **f '(0)** $= -3e^0\sin 0 + 2e^0\cos 0$ = **2**	2
$f''(x) = (-3e^{2x})(3\cos 3x) + (\sin 3x)(-6e^{2x}) + (2e^{2x})(-3\sin 3x) + (\cos 3x)(4e^{2x})$	
f ''(0) $= -9e^0\cos 0 - 6e^0\sin 0 - 6e^0\sin 0 + 4e^0\cos 0$ = - 9 – 0 – 0 + 4 = **- 5**	4

The Maclaurin's series is: $f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \ldots$	3
i.e. $\mathbf{e^{2x}\cos 3x = 1 + 2x - \frac{5}{2}x^2 + \ldots}$	**10**
Total:	

14. Show, using Maclaurin's series that the first four terms of the power series for cosh 2x is given by: $\cosh 2x = 1 + 2x^2 + \frac{2}{3}x^4 + \frac{4}{45}x^6$

	Marks
$f(x) = \cosh 2x$ $\quad$ $f(0) = \cosh 0 = 1$ $f'(x) = 2\sinh 2x$ $\quad$ $f'(0) = 2\sinh 0 = 0$ $f''(x) = 4\cosh 2x$ $\quad$ $f''(0) = 4$ $f'''(x) = 8\sinh 2x$ $\quad$ $f'''(0) = 0$ $f^{iv}(x) = 16\cosh 2x$ $\quad$ $f^{iv}(0) = 16$ $f^{v}(x) = 32\sinh 2x$ $\quad$ $f^{v}(0) = 0$ $f^{vi}(x) = 64\cosh 2x$ $\quad$ $f^{vi}(0) = 64$	6
Since $f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$ then $f(x) = \cosh 2x = 1 + x(0) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(16) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(64) + \ldots$	5
i.e. **cosh 2x =** $\mathbf{1 + 2x^2 + \frac{2}{3}x^4 + \frac{4}{45}x^6}$	**11**
Total:	

15. Expand the function $x^2\ln(1+\sin x)$ using Maclaurin's series and hence evaluate:

$\int_0^{1/2} x^2\ln(1+\sin x)\,dx$ correct to 2 significant figures.

	Marks
Let f(x) = ln(1 + sin x) **f(0)** = ln(1 + sin 0) = **0**	1
$f'(x) = \dfrac{\cos x}{1+\sin x}$ $\mathbf{f'(0)} = \dfrac{\cos 0}{1+\sin 0} = \mathbf{1}$	1
$f''(x) = \dfrac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{\left(1+\sin x\right)^2} = \dfrac{-\sin x-\sin^2 x-\cos^2 x}{\left(1+\sin x\right)^2}$	
$= \dfrac{-\sin x-1}{(1+\sin x)^2} = -\dfrac{(1+\sin x)}{(1+\sin x)^2} = \dfrac{-1}{1+\sin x}$ $\mathbf{f''(0)} = \dfrac{-1}{1+\sin 0} = \mathbf{-1}$	3 2
$f'''(x) = \dfrac{(1+\sin x)(0)-(-1)(\cos x)}{(1+\sin x)^2} = \dfrac{\cos x}{(1+\sin x)^2}$ $\mathbf{f'''(0)} = \dfrac{\cos 0}{(1+\sin 0)^2} = \mathbf{1}$	
Hence, ln(1 + sin x) = $f(0)+x\,f'(0)+\dfrac{x^2}{2!}f''(0)+\dfrac{x^3}{3!}f'''(0)+\ldots$	2
$= 0+x(1)+\dfrac{x^2}{2!}(-1)+\dfrac{x^3}{3!}(1)+\ldots = x-\dfrac{x^2}{2}+\dfrac{x^3}{6}-\ldots$	
Thus, $x^2\ln\left(2+\sin x\right) = x^2\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{6}-\ldots\right) = x^3-\dfrac{x^4}{2}+\dfrac{x^5}{6}-\ldots$	
and $\displaystyle\int_0^{1/2} x^2\ln(1+\sin x)\,dx = \int_0^{1/2}\left(x^3-\frac{x^4}{2}+\frac{x^5}{6}-\ldots\right)dx$	2
$= \left[\dfrac{x^4}{4}-\dfrac{x^5}{10}+\dfrac{x^6}{36}\right]_0^{1/2} = \left(\dfrac{\left(\frac{1}{2}\right)^4}{4}-\dfrac{\left(\frac{1}{2}\right)^5}{10}+\dfrac{\left(\frac{1}{2}\right)^6}{36}\right)-(0)$	2 **13**
= 0.015625 – 0.003125 + 0.000434	
= **0.013**, correct to 2 significant figures.	
Total:	

TOTAL MARKS FOR REVISION TEST 2: 100