

10 □ PARAMETRIC EQUATIONS AND POLAR COORDINATES

10.1 Curves Defined by Parametric Equations

1. $x = t^2 + t$, $y = 3^{t+1}$, $t = -2, -1, 0, 1, 2$

t	-2	-1	0	1	2
x	2	0	0	2	6
y	$\frac{1}{3}$	1	3	9	27

Therefore, the coordinates are $(2, \frac{1}{3})$, $(0, 1)$, $(0, 3)$, $(2, 9)$, and $(6, 27)$.

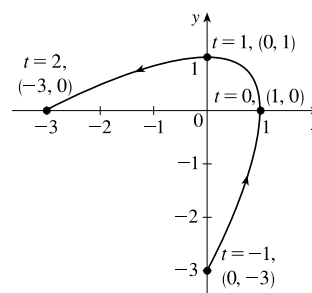
2. $x = \ln(t^2 + 1)$, $y = t/(t + 4)$, $t = -2, -1, 0, 1, 2$

t	-2	-1	0	1	2
x	$\ln 5$	$\ln 2$	0	$\ln 2$	$\ln 5$
y	-1	$-\frac{1}{3}$	0	$\frac{1}{5}$	$\frac{1}{3}$

Therefore, the coordinates are $(\ln 5, -1)$, $(\ln 2, -\frac{1}{3})$, $(0, 0)$, $(\ln 2, \frac{1}{5})$, and $(\ln 5, \frac{1}{3})$.

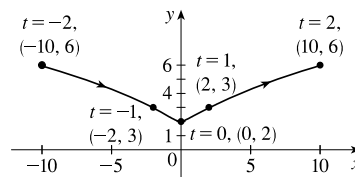
3. $x = 1 - t^2$, $y = 2t - t^2$, $-1 \leq t \leq 2$

t	-1	0	1	2
x	0	1	0	-3
y	-3	0	1	0



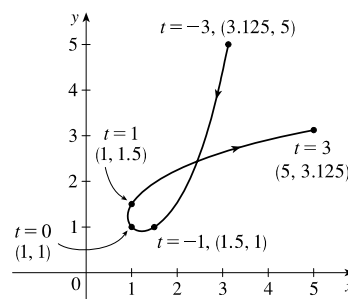
4. $x = t^3 + t$, $y = t^2 + 2$, $-2 \leq t \leq 2$

t	-2	-1	0	1	2
x	-10	-2	0	2	10
y	6	3	2	3	6



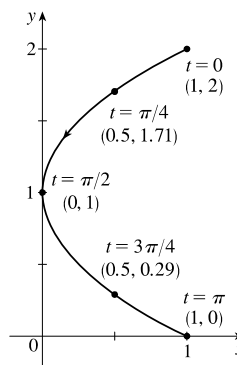
5. $x = 2^t - t$, $y = 2^{-t} + t$, $-3 \leq t \leq 3$

t	-3	-2	-1	0	1	2	3
x	3.125	2.25	1.5	1	1	2	5
y	5	2	1	1	1.5	2.25	3.125



6. $x = \cos^2 t$, $y = 1 + \cos t$, $0 \leq t \leq \pi$

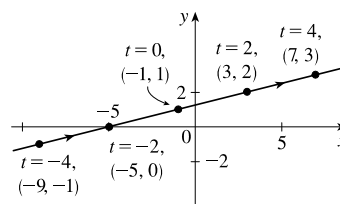
t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
x	1	0.5	0	0.5	1
y	2	1.707	1	0.293	0



7. $x = 2t - 1$, $y = \frac{1}{2}t + 1$

(a)

t	-4	-2	0	2	4
x	-9	-5	-1	3	7
y	-1	0	1	2	3



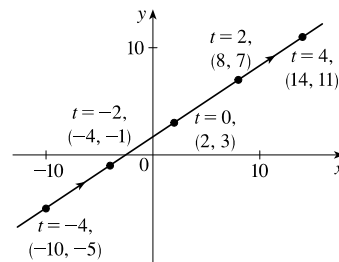
(b) $x = 2t - 1 \Rightarrow 2t = x + 1 \Rightarrow t = \frac{1}{2}x + \frac{1}{2}$, so

$$y = \frac{1}{2}t + 1 = \frac{1}{2}\left(\frac{1}{2}x + \frac{1}{2}\right) + 1 = \frac{1}{4}x + \frac{1}{4} + 1 \Rightarrow y = \frac{1}{4}x + \frac{5}{4}$$

8. $x = 3t + 2$, $y = 2t + 3$

(a)

t	-4	-2	0	2	4
x	-10	-4	2	8	14
y	-5	-1	3	7	11



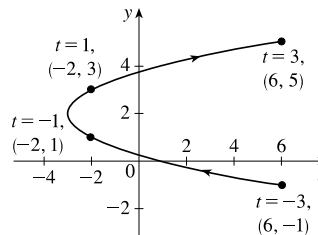
(b) $x = 3t + 2 \Rightarrow 3t = x - 2 \Rightarrow t = \frac{1}{3}x - \frac{2}{3}$, so

$$y = 2t + 3 = 2\left(\frac{1}{3}x - \frac{2}{3}\right) + 3 = \frac{2}{3}x - \frac{4}{3} + 3 \Rightarrow y = \frac{2}{3}x + \frac{5}{3}$$

9. $x = t^2 - 3$, $y = t + 2$, $-3 \leq t \leq 3$

(a)

t	-3	-1	1	3
x	6	-2	-2	6
y	-1	1	3	5



(b) $y = t + 2 \Rightarrow t = y - 2$, so

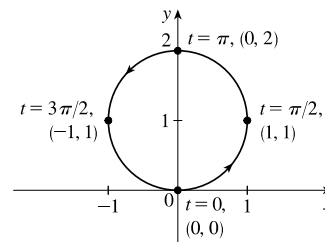
$$x = t^2 - 3 = (y - 2)^2 - 3 = y^2 - 4y + 4 - 3 \Rightarrow$$

$$x = y^2 - 4y + 1, -1 \leq y \leq 5$$

10. $x = \sin t, \quad y = 1 - \cos t, \quad 0 \leq t \leq 2\pi$

(a)

t	0	$\pi/2$	π	$3\pi/2$	2π
x	0	1	0	-1	0
y	0	1	2	1	0



(b) $x = \sin t, y = 1 - \cos t$ [or $y - 1 = -\cos t$] $\Rightarrow$

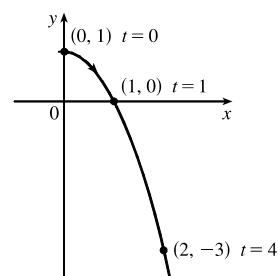
$$x^2 + (y - 1)^2 = (\sin t)^2 + (-\cos t)^2 \Rightarrow x^2 + (y - 1)^2 = 1.$$

 As t varies from 0 to 2π , the circle with center $(0, 1)$ and radius 1 is traced out.

11. $x = \sqrt{t}, \quad y = 1 - t$

(a)

t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	-1	-2	-3



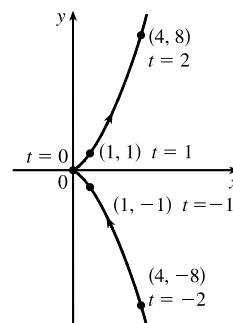
(b) $x = \sqrt{t} \Rightarrow t = x^2 \Rightarrow y = 1 - t = 1 - x^2$. Since $t \geq 0, x \geq 0$.

 So the curve is the right half of the parabola $y = 1 - x^2$.

12. $x = t^2, \quad y = t^3$

(a)

t	-2	-1	0	1	2
x	4	1	0	1	4
y	-8	-1	0	1	8



(b) $y = t^3 \Rightarrow t = \sqrt[3]{y} \Rightarrow x = t^2 = (\sqrt[3]{y})^2 = y^{2/3}$. $t \in \mathbb{R}, y \in \mathbb{R}, x \geq 0$.

13. (a) $x = 3 \cos t, \quad y = 3 \sin t, \quad 0 \leq t \leq \pi$

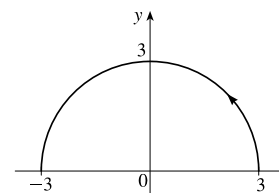
$$x^2 + y^2 = 9 \cos^2 t + 9 \sin^2 t = 9(\cos^2 t + \sin^2 t) = 9, \text{ which is the equation}$$

 of a circle with radius 3. For $0 \leq t \leq \pi/2$, we have $3 \geq x \geq 0$ and

 $0 \leq y \leq 3$. For $\pi/2 < t \leq \pi$, we have $0 > x \geq -3$ and $3 > y \geq 0$. Thus,

 the curve is the top half of the circle $x^2 + y^2 = 9$ traced counterclockwise.

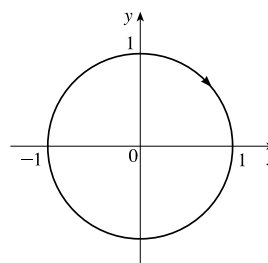
(b)



14. (a)
- $x = \sin 4\theta$
- ,
- $y = \cos 4\theta$
- ,
- $0 \leq \theta \leq \pi/2$

$x^2 + y^2 = \sin^2 4\theta + \cos^2 4\theta = 1$, which is the equation of a circle with radius 1. When $\theta = 0$, we have $x = 0$ and $y = 1$. For $0 \leq \theta \leq \pi/4$, we have $x \geq 0$. For $\pi/4 < \theta \leq \pi/2$, we have $x \leq 0$. Thus, the curve is the circle $x^2 + y^2 = 1$ traced clockwise starting at $(0, 1)$.

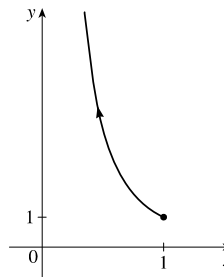
(b)



15. (a)
- $x = \cos \theta$
- ,
- $y = \sec^2 \theta$
- ,
- $0 \leq \theta < \pi/2$
- .

$y = \sec^2 \theta = \frac{1}{\cos^2 \theta} = \frac{1}{x^2}$. For $0 \leq \theta < \pi/2$, we have $1 \geq x > 0$ and $1 \leq y$.

(b)

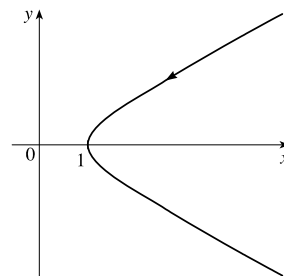


16. (a)
- $x = \csc t$
- ,
- $y = \cot t$
- ,
- $0 < t < \pi$

$y^2 - x^2 = \cot^2 t - \csc^2 t = 1$. For $0 < t < \pi$, we have $x > 1$.

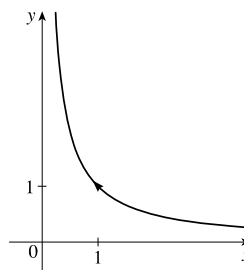
Thus, the curve is the right branch of the hyperbola $y^2 - x^2 = 1$.

(b)



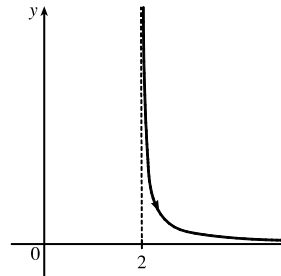
17. (a)
- $y = e^t = 1/e^{-t} = 1/x$
- for
- $x > 0$
- since
- $x = e^{-t}$
- . Thus, the curve is the portion of the hyperbola
- $y = 1/x$
- with
- $x > 0$
- .

(b)



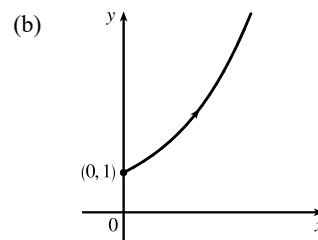
18. (a)
- $x = t + 2 \Rightarrow t = x - 2$
- .
- $y = 1/t = 1/(x - 2)$
- . For
- $t > 0$
- , we have
- $x > 2$
- and
- $y > 0$
- . Thus, the curve is the portion of the hyperbola
- $y = 1/(x - 2)$
- with
- $x > 2$
- .

(b)



19. (a) $x = \ln t, y = \sqrt{t}, t \geq 1$.

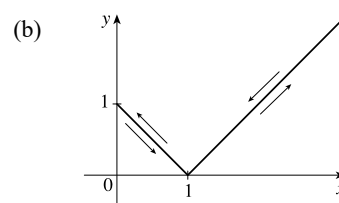
$$x = \ln t \Rightarrow t = e^x \Rightarrow y = \sqrt{t} = e^{x/2}, x \geq 0.$$



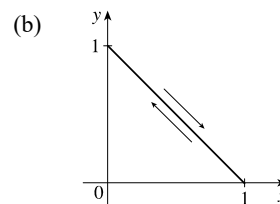
20. (a) $x = |t|, y = |1 - |t|| = |1 - x|$. For all t , we have $x \geq 0$ and

$y \geq 0$. Thus, the curve is the portion of the absolute value function

$$y = |1 - x| \text{ with } x \geq 0.$$



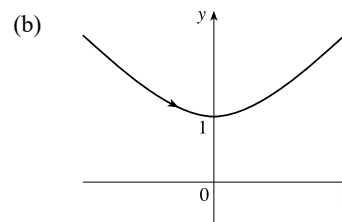
21. (a) $x = \sin^2 t, y = \cos^2 t, x + y = \sin^2 t + \cos^2 t = 1$. For all t , we have $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Thus, the curve is the portion of the line $x + y = 1$ or $y = -x + 1$ in the first quadrant.



22. (a) $x = \sinh t, y = \cosh t \Rightarrow y^2 - x^2 = \cosh^2 t - \sinh^2 t = 1$.

Since $y = \cosh t \geq 1$, we have the upper branch of the hyperbola

$$y^2 - x^2 = 1.$$



23. The parametric equations $x = 5 \cos t$ and $y = -5 \sin t$ both have period 2π . When $t = 0$, we have $x = 5$ and $y = 0$. When $t = \pi/2$, we have $x = 0$ and $y = -5$. This is one-fourth of a circle. Thus, the object completes one revolution in $4 \cdot \frac{\pi}{2} = 2\pi$ seconds following a clockwise path.

24. The parametric equations $x = 3 \sin\left(\frac{\pi}{4}t\right)$ and $y = 3 \cos\left(\frac{\pi}{4}t\right)$ both have period $\frac{2\pi}{\pi/4} = 8$. When $t = 0$, we have $x = 0$ and $y = 3$. When $t = 2$, we have $x = 3$ and $y = 0$. This is one-fourth of a circle. Thus, the object completes one revolution in $4 \cdot 2 = 8$ seconds following a clockwise path.

25. $x = 5 + 2 \cos \pi t, y = 3 + 2 \sin \pi t \Rightarrow \cos \pi t = \frac{x-5}{2}, \sin \pi t = \frac{y-3}{2}, \cos^2(\pi t) + \sin^2(\pi t) = 1 \Rightarrow$

$\left(\frac{x-5}{2}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$. The motion of the particle takes place on a circle centered at $(5, 3)$ with a radius 2. As t goes from 1 to 2, the particle starts at the point $(3, 3)$ and moves counterclockwise along the circle $\left(\frac{x-5}{2}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$ to $(7, 3)$ [one-half of a circle].

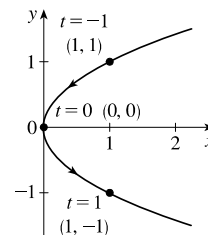
26. $x = 2 + \sin t, y = 1 + 3 \cos t \Rightarrow \sin t = x - 2, \cos t = \frac{y-1}{3}, \sin^2 t + \cos^2 t = 1 \Rightarrow (x-2)^2 + \left(\frac{y-1}{3}\right)^2 = 1$.

The motion of the particle takes place on an ellipse centered at $(2, 1)$. As t goes from $\pi/2$ to 2π , the particle starts at the point $(3, 1)$ and moves counterclockwise three-fourths of the way around the ellipse to $(2, 4)$.

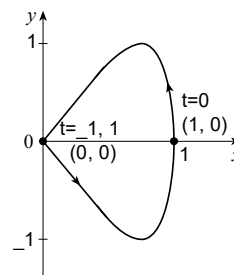
27. $x = 5 \sin t$, $y = 2 \cos t \Rightarrow \sin t = \frac{x}{5}$, $\cos t = \frac{y}{2}$. $\sin^2 t + \cos^2 t = 1 \Rightarrow \left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$. The motion of the particle takes place on an ellipse centered at $(0, 0)$. As t goes from $-\pi$ to 5π , the particle starts at the point $(0, -2)$ and moves clockwise around the ellipse 3 times.
28. $y = \cos^2 t = 1 - \sin^2 t = 1 - x^2$. The motion of the particle takes place on the parabola $y = 1 - x^2$. As t goes from $-\pi$ to 0 , the particle starts at the point $(0, 1)$, moves to $(1, 0)$, and goes back to $(0, 1)$. As t goes from 0 to 2π , the particle moves to $(-1, 0)$ and goes back to $(0, 1)$. The particle repeats this motion as t goes from 0 to 2π .
29. We must have $1 \leq x \leq 4$ and $2 \leq y \leq 3$. So the graph of the curve must be contained in the rectangle $[1, 4]$ by $[2, 3]$.
30. (a) From the first graph, we have $1 \leq x \leq 2$. From the second graph, we have $-1 \leq y \leq 1$. The only choice that satisfies either of those conditions is III.
- (b) From the first graph, the values of x cycle through the values from -2 to 2 four times. From the second graph, the values of y cycle through the values from -2 to 2 six times. Choice I satisfies these conditions.
- (c) From the first graph, the values of x cycle through the values from -2 to 2 three times. From the second graph, we have $0 \leq y \leq 2$. Choice IV satisfies these conditions.
- (d) From the first graph, the values of x cycle through the values from -2 to 2 two times. From the second graph, the values of y do the same thing. Choice II satisfies these conditions.

31. When $t = -1$, $(x, y) = (1, 1)$. As t increases to 0 , x and y both decrease to 0 .

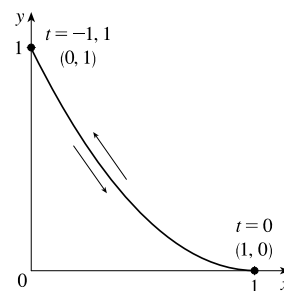
As t increases from 0 to 1 , x increases from 0 to 1 and y decreases from 0 to -1 . As t increases beyond 1 , x continues to increase and y continues to decrease. For $t < -1$, x and y are both positive and decreasing. We could achieve greater accuracy by estimating x - and y -values for selected values of t from the given graphs and plotting the corresponding points.



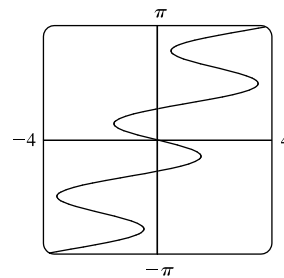
32. When $t = -1$, $(x, y) = (0, 0)$. As t increases to 0 , x increases from 0 to 1 , while y first decreases to -1 and then increases to 0 . As t increases from 0 to 1 , x decreases from 1 to 0 , while y first increases to 1 and then decreases to 0 . We could achieve greater accuracy by estimating x - and y -values for selected values of t from the given graphs and plotting the corresponding points.



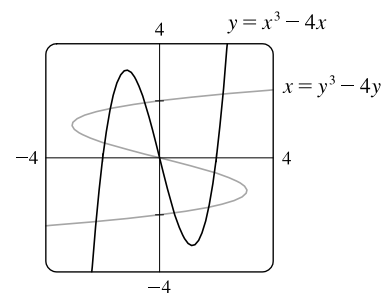
33. When $t = -1$, $(x, y) = (0, 1)$. As t increases to 0 , x increases from 0 to 1 and y decreases from 1 to 0 . As t increases from 0 to 1 , the curve is retraced in the opposite direction with x decreasing from 1 to 0 and y increasing from 0 to 1 . We could achieve greater accuracy by estimating x - and y -values for selected values of t from the given graphs and plotting the corresponding points.



34. (a) $x = t^4 - t + 1 = (t^4 + 1) - t > 0$ [think of the graphs of $y = t^4 + 1$ and $y = t$] and $y = t^2 \geq 0$, so these equations are matched with graph V.
- (b) $y = \sqrt{t} \geq 0$. $x = t^2 - 2t = t(t - 2)$ is negative for $0 < t < 2$, so these equations are matched with graph I.
- (c) $x = t^3 - 2t = t(t^2 - 2) = t(t + \sqrt{2})(t - \sqrt{2})$, $y = t^2 - t = t(t - 1)$. The equation $x = 0$ has three solutions and the equation $y = 0$ has two solutions. Thus, the curve has three y -intercepts and two x -intercepts, which matches graph II.
Alternate method: $x = t^3 - 2t$, $y = t^2 - t = (t^2 - t + \frac{1}{4}) - \frac{1}{4} = (t - \frac{1}{2})^2 - \frac{1}{4}$ so $y \geq -\frac{1}{4}$ on this curve, whereas x is unbounded. These equations are matched with graph II.
- (d) $x = \cos 5t$ has period $2\pi/5$ and $y = \sin 2t$ has period π , so x will take on the values -1 to 1 , and then 1 to -1 , before y takes on the values -1 to 1 . Note that when $t = 0$, $(x, y) = (1, 0)$. These equations are matched with graph VI.
- (e) $x = t + \sin 4t$, $y = t^2 + \cos 3t$. As t becomes large, t and t^2 become the dominant terms in the expressions for x and y , so the graph will look like the graph of $y = x^2$, but with oscillations. These equations are matched with graph IV.
- (f) $x = t + \sin 2t$, $y = t + \sin 3t$. As t becomes large, t becomes the dominant term in the expressions for both x and y , so the graph will look like the graph of $y = x$, but with oscillations. These equations are matched with graph III.
35. Use $y = t$ and $x = t - 2 \sin \pi t$ with a t -interval of $[-\pi, \pi]$.



36. Use $x_1 = t$, $y_1 = t^3 - 4t$ and $x_2 = t^3 - 4t$, $y_2 = t$ with a t -interval of $[-3, 3]$. There are 9 points of intersection; $(0, 0)$ is fairly obvious. The point in quadrant I is approximately $(2.2, 2.2)$, and by symmetry, the point in quadrant III is approximately $(-2.2, -2.2)$. The other six points are approximately $(\pm 1.9, \pm 0.5)$, $(\pm 1.7, \pm 1.7)$, and $(\pm 0.5, \pm 1.9)$.



37. (a) $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$, $0 \leq t \leq 1$. Clearly the curve passes through $P_1(x_1, y_1)$ when $t = 0$ and through $P_2(x_2, y_2)$ when $t = 1$. For $0 < t < 1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2 . For every value of t , x and y satisfy the relation $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$, which is the equation of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Finally, any point (x, y) on that line satisfies $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$; if we call that common value t , then the given parametric equations yield the point (x, y) ; and any (x, y) on the line between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ yields a value of t in $[0, 1]$. So the given parametric equations exactly specify the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$.

(b) $x = -2 + [3 - (-2)]t = -2 + 5t$ and $y = 7 + (-1 - 7)t = 7 - 8t$ for $0 \leq t \leq 1$.

38. For the side of the triangle from A to B , use $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (4, 2)$.

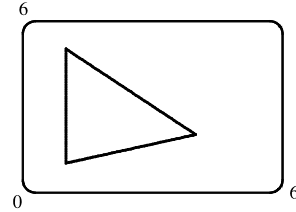
Hence, the equations are

$$x = x_1 + (x_2 - x_1)t = 1 + (4 - 1)t = 1 + 3t,$$

$$y = y_1 + (y_2 - y_1)t = 1 + (2 - 1)t = 1 + t.$$

Graphing $x = 1 + 3t$ and $y = 1 + t$ with $0 \leq t \leq 1$ gives us the side of the

triangle from A to B . Similarly, for the side BC we use $x = 4 - 3t$ and $y = 2 + 3t$, and for the side AC we use $x = 1$ and $y = 1 + 4t$.



39. The result in Example 4 indicates the parametric equations have the form $x = h + r \sin bt$ and $y = k + r \cos bt$ where (h, k) is the center of the circle with radius r and $b = 2\pi/\text{period}$. (The use of positive sine in the x -equation and positive cosine in the y -equation results in a clockwise motion.) With $h = 0, k = 0$ and $b = 2\pi/4\pi = 1/2$, we have $x = 5 \sin(\frac{1}{2}t)$, $y = 5 \cos(\frac{1}{2}t)$.
40. As in Example 4, we use parametric equations of the form $x = h + r \cos bt$ and $y = k + r \sin bt$ where $(h, k) = (1, 3)$ is the center of the circle with radius $r = 1$ and $b = 2\pi/\text{period} = 2\pi/3$. (The use of positive cosine in the x -equation and positive sine in the y -equation results in a counterclockwise motion.) Thus, $x = 1 + \cos(\frac{2\pi}{3}t)$, $y = 3 + \sin(\frac{2\pi}{3}t)$.

41. The circle $x^2 + (y - 1)^2 = 4$ has center $(0, 1)$ and radius 2, so by Example 4 it can be represented by $x = 2 \cos t$, $y = 1 + 2 \sin t$, $0 \leq t \leq 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2, 1)$.
- (a) To get a clockwise orientation, we could change the equations to $x = 2 \cos t$, $y = 1 - 2 \sin t$, $0 \leq t \leq 2\pi$.
- (b) To get three times around in the counterclockwise direction, we use the original equations $x = 2 \cos t$, $y = 1 + 2 \sin t$ with the domain expanded to $0 \leq t \leq 6\pi$.
- (c) To start at $(0, 3)$ using the original equations, we must have $x_1 = 0$; that is, $2 \cos t = 0$. Hence, $t = \frac{\pi}{2}$. So we use

$$x = 2 \cos t, y = 1 + 2 \sin t, \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}.$$

Alternatively, if we want t to start at 0, we could change the equations of the curve. For example, we could use

$$x = -2 \sin t, y = 1 + 2 \cos t, 0 \leq t \leq \pi.$$

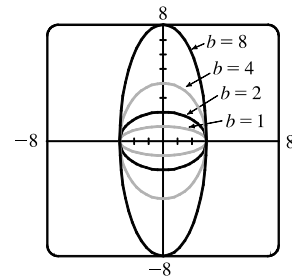
42. (a) Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a \sin t$ and

$$y = b \cos t \text{ with } 0 \leq t \leq 2\pi \text{ as possible parametric equations for the ellipse}$$

$$x^2/a^2 + y^2/b^2 = 1.$$

- (b) The equations are $x = 3 \sin t$ and $y = b \cos t$ for $b \in \{1, 2, 4, 8\}$.

- (c) As b increases, the ellipse stretches vertically.



43. *Big circle:* It's centered at $(2, 2)$ with a radius of 2, so by Example 4, parametric equations are

$$x = 2 + 2 \cos t, \quad y = 2 + 2 \sin t, \quad 0 \leq t \leq 2\pi$$

Small circles: They are centered at $(1, 3)$ and $(3, 3)$ with a radius of 0.1. By Example 4, parametric equations are

$$(left) \quad x = 1 + 0.1 \cos t, \quad y = 3 + 0.1 \sin t, \quad 0 \leq t \leq 2\pi$$

and

$$(right) \quad x = 3 + 0.1 \cos t, \quad y = 3 + 0.1 \sin t, \quad 0 \leq t \leq 2\pi$$

Semicircle: It's the lower half of a circle centered at $(2, 2)$ with radius 1. By Example 4, parametric equations are

$$x = 2 + 1 \cos t, \quad y = 2 + 1 \sin t, \quad \pi \leq t \leq 2\pi$$

To get all four graphs on the same screen with a typical graphing calculator, we need to change the last t -interval to $[0, 2\pi]$ in order to match the others. We can do this by changing t to $0.5t$. This change gives us the upper half. There are several ways to get the lower half—one is to change the “+” to a “−” in the y -assignment, giving us

$$x = 2 + 1 \cos(0.5t), \quad y = 2 - 1 \sin(0.5t), \quad 0 \leq t \leq 2\pi$$

44. If you are using a calculator or computer that can overlay graphs (using multiple t -intervals), the following is appropriate.

Left side: $x = 1$ and y goes from 1.5 to 4, so use

$$x = 1, \quad y = t, \quad 1.5 \leq t \leq 4$$

Right side: $x = 10$ and y goes from 1.5 to 4, so use

$$x = 10, \quad y = t, \quad 1.5 \leq t \leq 4$$

Bottom: x goes from 1 to 10 and $y = 1.5$, so use

$$x = t, \quad y = 1.5, \quad 1 \leq t \leq 10$$

Handle: It starts at $(10, 4)$ and ends at $(13, 7)$, so use

$$x = 10 + t, \quad y = 4 + t, \quad 0 \leq t \leq 3$$

Left wheel: It's centered at $(3, 1)$, has a radius of 1, and appears to go about 30° above the horizontal, so use

$$x = 3 + 1 \cos t, \quad y = 1 + 1 \sin t, \quad \frac{5\pi}{6} \leq t \leq \frac{13\pi}{6}$$

Right wheel: Similar to the left wheel with center $(8, 1)$, so use

$$x = 8 + 1 \cos t, \quad y = 1 + 1 \sin t, \quad \frac{5\pi}{6} \leq t \leq \frac{13\pi}{6}$$

If you are using a calculator or computer that cannot overlay graphs (using one t -interval), the following is appropriate.

We'll start by picking the t -interval $[0, 2.5]$ since it easily matches the t -values for the two sides. We now need to find parametric equations for all graphs with $0 \leq t \leq 2.5$.

Left side: $x = 1$ and y goes from 1.5 to 4, so use

$$x = 1, \quad y = 1.5 + t, \quad 0 \leq t \leq 2.5$$

[continued]

Right side: $x = 10$ and y goes from 1.5 to 4, so use

$$x = 10, \quad y = 1.5 + t, \quad 0 \leq t \leq 2.5$$

Bottom: x goes from 1 to 10 and $y = 1.5$, so use

$$x = 1 + 3.6t, \quad y = 1.5, \quad 0 \leq t \leq 2.5$$

To get the x -assignment, think of creating a linear function such that when $t = 0$, $x = 1$ and when $t = 2.5$, $x = 10$. We can use the point-slope form of a line with $(t_1, x_1) = (0, 1)$ and $(t_2, x_2) = (2.5, 10)$.

$$x - 1 = \frac{10 - 1}{2.5 - 0}(t - 0) \Rightarrow x = 1 + 3.6t.$$

Handle: It starts at $(10, 4)$ and ends at $(13, 7)$, so use

$$x = 10 + 1.2t, \quad y = 4 + 1.2t, \quad 0 \leq t \leq 2.5$$

$$(t_1, x_1) = (0, 10) \text{ and } (t_2, x_2) = (2.5, 13) \text{ gives us } x - 10 = \frac{13 - 10}{2.5 - 0}(t - 0) \Rightarrow x = 10 + 1.2t.$$

$$(t_1, y_1) = (0, 4) \text{ and } (t_2, y_2) = (2.5, 7) \text{ gives us } y - 4 = \frac{7 - 4}{2.5 - 0}(t - 0) \Rightarrow y = 4 + 1.2t.$$

Left wheel: It's centered at $(3, 1)$, has a radius of 1, and appears to go about 30° above the horizontal, so use

$$x = 3 + 1 \cos\left(\frac{8\pi}{15}t + \frac{5\pi}{6}\right), \quad y = 1 + 1 \sin\left(\frac{8\pi}{15}t + \frac{5\pi}{6}\right), \quad 0 \leq t \leq 2.5$$

$$(t_1, \theta_1) = \left(0, \frac{5\pi}{6}\right) \text{ and } (t_2, \theta_2) = \left(\frac{5}{2}, \frac{13\pi}{6}\right) \text{ gives us } \theta - \frac{5\pi}{6} = \frac{\frac{13\pi}{6} - \frac{5\pi}{6}}{\frac{5}{2} - 0}(t - 0) \Rightarrow \theta = \frac{5\pi}{6} + \frac{8\pi}{15}t.$$

Right wheel: Similar to the left wheel with center $(8, 1)$, so use

$$x = 8 + 1 \cos\left(\frac{8\pi}{15}t + \frac{5\pi}{6}\right), \quad y = 1 + 1 \sin\left(\frac{8\pi}{15}t + \frac{5\pi}{6}\right), \quad 0 \leq t \leq 2.5$$

45. (a) (i) $x = t^2, y = t \Rightarrow y^2 = t^2 = x$

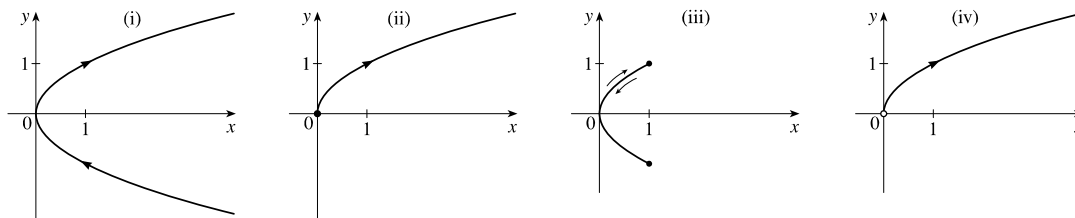
(ii) $x = t, y = \sqrt{t} \Rightarrow y^2 = t = x$

(iii) $x = \cos^2 t, y = \cos t \Rightarrow y^2 = \cos^2 t = x$

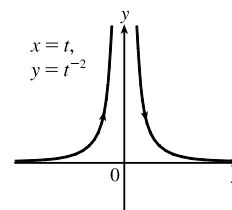
(iv) $x = 3^{2t}, y = 3^t \Rightarrow y^2 = (3^t)^2 = 3^{2t} = x.$

Thus, the points on all four of the given parametric curves satisfy the Cartesian equation $y^2 = x$.

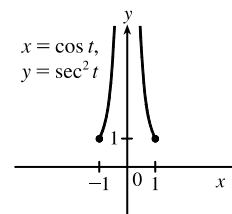
(b) The graph of $y^2 = x$ is a right-opening parabola with vertex at the origin. For curve (i), $x \geq 0$ and y is unbounded so the graph contains the entire parabola. For (ii), $y = \sqrt{t}$ requires that $t \geq 0$, so that both $x \geq 0$ and $y \geq 0$, which captures the upper half of the parabola, including the origin. For (iii), $-1 \leq \cos t \leq 1$ so the graph is the portion of the parabola contained in the intervals $0 \leq x \leq 1$ and $-1 \leq y \leq 1$. For (iv), $x > 0$ and $y > 0$, which captures the upper half of the parabola excluding the origin.



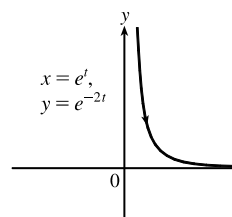
46. (a) $x = t$, so $y = t^{-2} = x^{-2}$. We get the entire curve $y = 1/x^2$ traversed in a left-to-right direction.



- (b) $x = \cos t$, $y = \sec^2 t = \frac{1}{\cos^2 t} = \frac{1}{x^2}$. Since $\sec t \geq 1$, we only get the parts of the curve $y = 1/x^2$ with $y \geq 1$. We get the first quadrant portion of the curve when $x > 0$, that is, $\cos t > 0$, and we get the second quadrant portion of the curve when $x < 0$, that is, $\cos t < 0$.

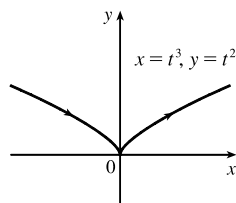


- (c) $x = e^t$, $y = e^{-2t} = (e^t)^{-2} = x^{-2}$. Since e^t and e^{-2t} are both positive, we only get the first quadrant portion of the curve $y = 1/x^2$.



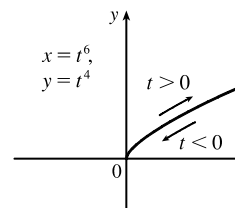
47. (a) $x = t^3 \Rightarrow t = x^{1/3}$, so $y = t^2 = x^{2/3}$.

We get the entire curve $y = x^{2/3}$ traversed in a left to right direction.



- (b) $x = t^6 \Rightarrow t = x^{1/6}$, so $y = t^4 = x^{4/6} = x^{2/3}$.

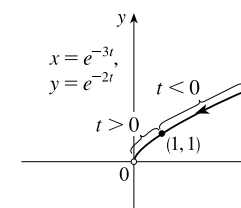
Since $x = t^6 \geq 0$, we only get the right half of the curve $y = x^{2/3}$.



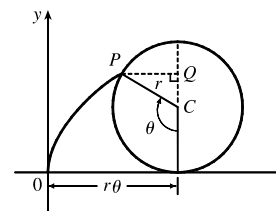
- (c) $x = e^{-3t} = (e^{-t})^3$ [so $e^{-t} = x^{1/3}$],

$$y = e^{-2t} = (e^{-t})^2 = (x^{1/3})^2 = x^{2/3}.$$

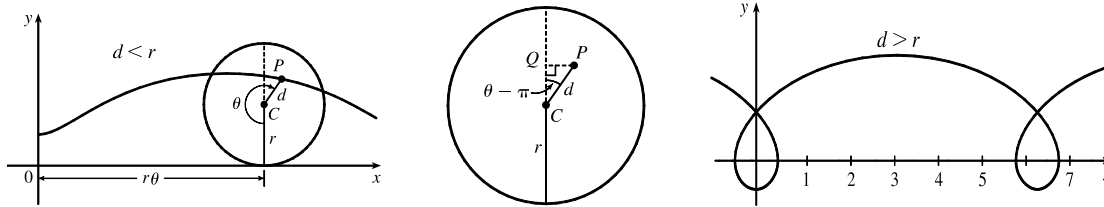
If $t < 0$, then x and y are both larger than 1. If $t > 0$, then x and y are between 0 and 1. Since $x > 0$ and $y > 0$, the curve never quite reaches the origin.



48. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as in Example 7, and Q has coordinates $(r\theta, r + r \cos(\pi - \theta)) = (r\theta, r(1 - \cos \theta))$ [since $\cos(\pi - \alpha) = \cos \pi \cos \alpha + \sin \pi \sin \alpha = -\cos \alpha$], so P has coordinates $(r\theta - r \sin(\pi - \theta), r(1 - \cos \theta)) = (r(\theta - \sin \theta), r(1 - \cos \theta))$ [since $\sin(\pi - \alpha) = \sin \pi \cos \alpha - \cos \pi \sin \alpha = \sin \alpha$]. Again we have the parametric equations $x = r(\theta - \sin \theta)$, $y = r(1 - \cos \theta)$.

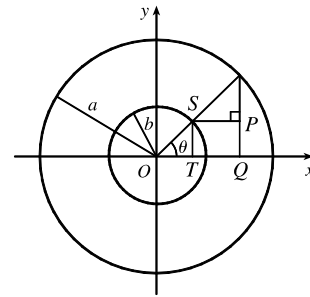


49. The first two diagrams depict the case $\pi < \theta < \frac{3\pi}{2}$, $d < r$. As in Example 7, C has coordinates $(r\theta, r)$. Now Q (in the second diagram) has coordinates $(r\theta, r + d\cos(\theta - \pi)) = (r\theta, r - d\cos\theta)$, so a typical point P of the trochoid has coordinates $(r\theta + d\sin(\theta - \pi), r - d\cos\theta)$. That is, P has coordinates (x, y) , where $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$. When $d = r$, these equations agree with those of the cycloid.



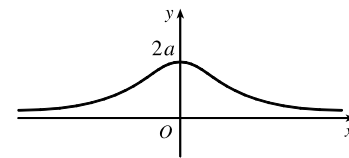
50. In polar coordinates, an equation for the circle is $r = 2a \sin \theta$. Thus, the coordinates of Q are $x = r \cos \theta = 2a \sin \theta \cos \theta$ and $y = r \sin \theta = 2a \sin^2 \theta$. The coordinates of R are $x = 2a \cot \theta$ and $y = 2a$. Since P is the midpoint of QR , we use the midpoint formula to get $x = a(\sin \theta \cos \theta + \cot \theta)$ and $y = a(1 + \sin^2 \theta)$.

51. It is apparent that $x = |OQ|$ and $y = |QP| = |ST|$. From the diagram, $x = |OQ| = a \cos \theta$ and $y = |ST| = b \sin \theta$. Thus, the parametric equations are $x = a \cos \theta$ and $y = b \sin \theta$. To eliminate θ we rearrange: $\sin \theta = y/b \Rightarrow \sin^2 \theta = (y/b)^2$ and $\cos \theta = x/a \Rightarrow \cos^2 \theta = (x/a)^2$. Adding the two equations: $\sin^2 \theta + \cos^2 \theta = 1 = x^2/a^2 + y^2/b^2$. Thus, we have an ellipse.



52. A has coordinates $(a \cos \theta, a \sin \theta)$. Since OA is perpendicular to AB , $\triangle OAB$ is a right triangle and B has coordinates $(a \sec \theta, 0)$. It follows that P has coordinates $(a \sec \theta, b \sin \theta)$. Thus, the parametric equations are $x = a \sec \theta$, $y = b \sin \theta$.

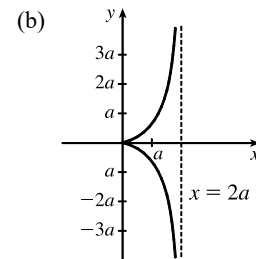
53. $C = (2a \cot \theta, 2a)$, so the x -coordinate of P is $x = 2a \cot \theta$. Let $B = (0, 2a)$. Then $\angle OAB$ is a right angle and $\angle OBA = \theta$, so $|OA| = 2a \sin \theta$ and $A = ((2a \sin \theta) \cos \theta, (2a \sin \theta) \sin \theta)$. Thus, the y -coordinate of P is $y = 2a \sin^2 \theta$.



54. (a) Let θ be the angle of inclination of segment OP . Then $|OB| = \frac{2a}{\cos \theta}$. Let $C = (2a, 0)$. Then by use of right triangle OAC we see that $|OA| = 2a \cos \theta$. Now

$$\begin{aligned} |OP| &= |AB| = |OB| - |OA| \\ &= 2a \left(\frac{1}{\cos \theta} - \cos \theta \right) = 2a \frac{1 - \cos^2 \theta}{\cos \theta} = 2a \frac{\sin^2 \theta}{\cos \theta} = 2a \sin \theta \tan \theta \end{aligned}$$

So P has coordinates $x = 2a \sin \theta \tan \theta \cdot \cos \theta = 2a \sin^2 \theta$ and $y = 2a \sin \theta \tan \theta \cdot \sin \theta = 2a \sin^2 \theta \tan \theta$.



55. (a) *Red particle:* $x = t + 5$, $y = t^2 + 4t + 6$

Blue particle: $x = 2t + 1$, $y = 2t + 6$

Substituting $x = 1$ and $y = 6$ into the parametric equations for the red particle gives $1 = t + 5$ and $6 = t^2 + 4t + 6$, which are both satisfied when $t = -4$. Making the same substitution for the blue particle gives $1 = 2t + 1$ and $6 = 2t + 6$, which are both satisfied when $t = 0$. Repeating the process for $x = 6$ and $y = 11$, the red particle's equations become $6 = t + 5$ and $11 = t^2 + 4t + 6$, which are both satisfied when $t = 1$. Similarly, the blue particle's equations become $6 = 2t + 1$ and $11 = 2t + 6$, which are both satisfied when $t = 2.5$. Thus, $(1, 6)$ and $(6, 11)$ are both intersection points, but they are not collision points, since the particles reach each of these points at different times.

- (b) *Blue particle:* $x = 2t + 1 \Rightarrow t = \frac{1}{2}(x - 1)$.

Substituting into the equation for y gives $y = 2t + 6 = 2\left[\frac{1}{2}(x - 1)\right] + 6 = x + 5$.

Green particle: $x = 2t + 4 \Rightarrow t = \frac{1}{2}(x - 4)$.

Substituting into the equation for y gives $y = 2t + 9 = 2\left[\frac{1}{2}(x - 4)\right] + 9 = x + 5$.

Thus, the green and blue particles both move along the line $y = x + 5$.

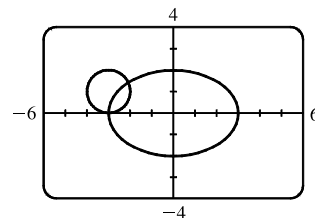
Now, the red and green particles will collide if there is a time t when both particles are at the same point. Equating the x parametric equations, we find $t + 5 = 2t + 4$, which is satisfied when $t = 1$, and gives $x = 1 + 5 = 6$. Substituting $t = 1$ into the red and green particles' y equations gives $y = (1)^2 + 4(1) + 6 = 11$ and $y = 2(1) + 9 = 11$, respectively. Thus, the red and green particles collide at the point $(6, 11)$ when $t = 1$.

56. (a) $x = 3 \sin t$, $y = 2 \cos t$, $0 \leq t \leq 2\pi$;

$x = -3 + \cos t$, $y = 1 + \sin t$, $0 \leq t \leq 2\pi$

There are 2 points of intersection:

$(-3, 0)$ and approximately $(-2.1, 1.4)$.



- (b) A collision point occurs when $x_1 = x_2$ and $y_1 = y_2$ for the same t . So solve the equations:

$$3 \sin t = -3 + \cos t \quad (1)$$

$$2 \cos t = 1 + \sin t \quad (2)$$

From (2), $\sin t = 2 \cos t - 1$. Substituting into (1), we get $3(2 \cos t - 1) = -3 + \cos t \Rightarrow 5 \cos t = 0 \quad (*) \Rightarrow$

$\cos t = 0 \Rightarrow t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. We check that $t = \frac{3\pi}{2}$ satisfies (1) and (2) but $t = \frac{\pi}{2}$ does not. So the only collision point occurs when $t = \frac{3\pi}{2}$, and this gives the point $(-3, 0)$. [We could check our work by graphing x_1 and x_2 together as functions of t and, on another plot, y_1 and y_2 as functions of t . If we do so, we see that the only value of t for which *both* pairs of graphs intersect is $t = \frac{3\pi}{2}$.]

- (c) The circle is centered at $(3, 1)$ instead of $(-3, 1)$. There are still 2 intersection points: $(3, 0)$ and $(2.1, 1.4)$, but there are no collision points, since $(*)$ in part (b) becomes $5 \cos t = 6 \Rightarrow \cos t = \frac{6}{5} > 1$.

57. (a) $x = 1 - t^2$, $y = t - t^3$. The curve intersects itself if there are two distinct times $t = a$ and $t = b$ (with $a < b$) such that $x(a) = x(b)$ and $y(a) = y(b)$. The equation $x(a) = x(b)$ gives $1 - a^2 = 1 - b^2$ so that $a^2 = b^2$. Since $a \neq b$ by assumption, we must have $a = -b$. Substituting into the equation for y gives $y(-b) = y(b) \Rightarrow$
 $-b - (-b)^3 = b - b^3 \Rightarrow 2b^3 - 2b = 0 \Rightarrow 2b(b-1)(b+1) = 0 \Rightarrow b = -1, 0, 1$. Since $a < b$, the only valid solution is $b = 1$, which corresponds to $a = -1$ and results in the coordinates $x = 0$ and $y = 0$. Thus, the curve intersects itself at $(0, 0)$ when $t = -1$ and $t = 1$.

- (b) $x = 2t - t^3$, $y = t - t^2$. Similar to part (a), we try to find the times $t = a$ and $t = b$ with $a < b$ such that $x(a) = x(b)$ and $y(a) = y(b)$. The equation $y(a) = y(b)$ gives $a - a^2 = b - b^2 \Rightarrow 0 = a^2 - a + (b - b^2)$. Using the quadratic formula to solve for a , we get

$$a = \frac{1 \pm \sqrt{1 - 4(b - b^2)}}{2} = \frac{1 \pm \sqrt{4b^2 - 4b + 1}}{2} = \frac{1 \pm \sqrt{(2b-1)^2}}{2} = \frac{1 \pm (2b-1)}{2} \Rightarrow a = b \text{ or } a = 1 - b.$$

Since $a < b$ by assumption, we reject the first solution and substitute $a = 1 - b$ into $x(a) = x(b) \Rightarrow x(1 - b) = x(b) \Rightarrow$
 $2(1 - b) - (1 - b)^3 = 2b - b^3$. Expanding and simplifying gives $2b^3 - 3b^2 - b + 1 = 0$. By graphing the equation, we see that $b = \frac{1}{2}$ is a zero, so $2b - 1$ is a factor, and by long division $b^2 - b - 1$ is another factor. Hence, the solutions are

$b = \frac{1}{2}$ and $b = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$ (found using the quadratic formula). Since $a = 1 - b$ and we require $a < b$, the only valid

solution is $b = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, which corresponds to $a = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ and results in the coordinates

$x = 2(\frac{1}{2} - \frac{1}{2}\sqrt{5}) - (\frac{1}{2} - \frac{1}{2}\sqrt{5})^3 = -1$ and $y = \frac{1}{2} - \frac{1}{2}\sqrt{5} - (\frac{1}{2} - \frac{1}{2}\sqrt{5})^2 = -1$. Thus, the curve intersects itself at $(-1, -1)$ when $t = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ and $t = \frac{1}{2} + \frac{1}{2}\sqrt{5}$.

58. (a) If $\alpha = 30^\circ$ and $v_0 = 500$ m/s, then the equations become $x = (500 \cos 30^\circ)t = 250\sqrt{3}t$ and

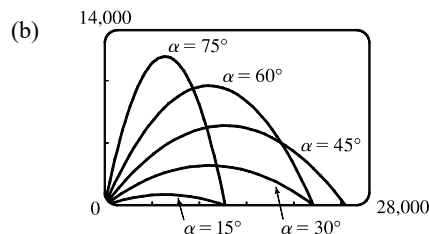
$$y = (500 \sin 30^\circ)t - \frac{1}{2}(9.8)t^2 = 250t - 4.9t^2. \quad y = 0 \text{ when } t = 0 \text{ (when the gun is fired) and again when}$$

$$t = \frac{250}{4.9} \approx 51 \text{ s. Then } x = (250\sqrt{3})\left(\frac{250}{4.9}\right) \approx 22,092 \text{ m, so the bullet hits the ground about 22 km from the gun.}$$

The formula for y is quadratic in t . To find the maximum y -value, we will complete the square:

$$y = -4.9\left(t^2 - \frac{250}{4.9}t\right) = -4.9\left[t^2 - \frac{250}{4.9}t + \left(\frac{125}{4.9}\right)^2\right] + \frac{125^2}{4.9} = -4.9\left(t - \frac{125}{4.9}\right)^2 + \frac{125^2}{4.9} \leq \frac{125^2}{4.9}$$

with equality when $t = \frac{125}{4.9}$ s, so the maximum height attained is $\frac{125^2}{4.9} \approx 3189$ m.



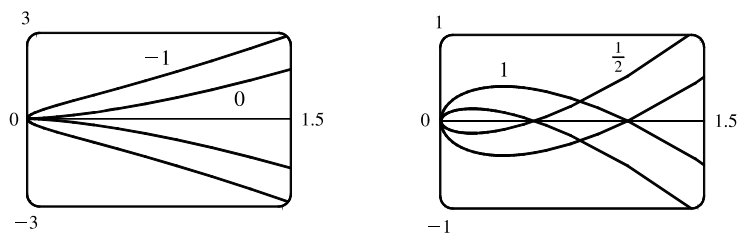
As α ($0^\circ < \alpha < 90^\circ$) increases up to 45° , the projectile attains a greater height and a greater range. As α increases past 45° , the projectile attains a greater height, but its range decreases.

$$(c) \quad x = (v_0 \cos \alpha)t \Rightarrow t = \frac{x}{v_0 \cos \alpha}.$$

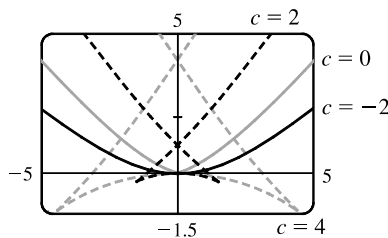
$$y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y = (v_0 \sin \alpha) \frac{x}{v_0 \cos \alpha} - \frac{g}{2} \left(\frac{x}{v_0 \cos \alpha} \right)^2 = (\tan \alpha)x - \left(\frac{g}{2v_0^2 \cos^2 \alpha} \right) x^2,$$

which is the equation of a parabola (quadratic in x).

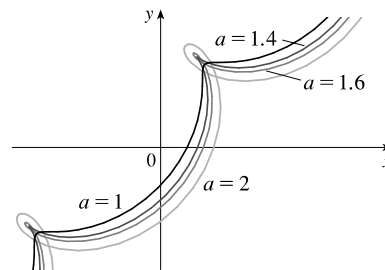
59. $x = t^2, y = t^3 - ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \leq t \leq \pi$. Note that all the members of the family are symmetric about the x -axis. For $c < 0$, the graph does not cross itself, but for $c = 0$ it has a cusp at $(0, 0)$ and for $c > 0$ the graph crosses itself at $x = c$, so the loop grows larger as c increases.



60. $x = 2ct - 4t^3, y = -ct^2 + 3t^4$. We use a graphing device to produce the graphs for various values of c with $-\pi \leq t \leq \pi$. Note that all the members of the family are symmetric about the y -axis. When $c < 0$, the graph resembles that of a polynomial of even degree, but when $c = 0$ there is a corner at the origin, and when $c > 0$, the graph crosses itself at the origin, and has two cusps below the x -axis. The size of the “swallowtail” increases as c increases.

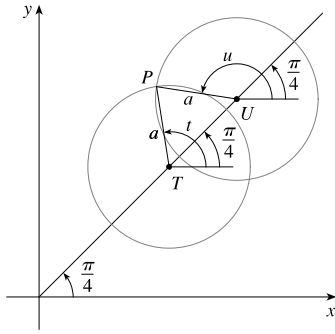


61. $x = t + a \cos t, y = t + a \sin t, a > 0$. From the first figure, we see that curves roughly follow the line $y = x$, and they start having loops when a is between 1.4 and 1.6. The loops increase in size as a increases.



While not required, the following is a solution to determine the *exact* values for which the curve has a loop, that is, we seek the values of a for which there exist parameter values t and u such that $t < u$ and $(t + a \cos t, t + a \sin t) = (u + a \cos u, u + a \sin u)$.

[continued]



In the diagram at the left, T denotes the point (t, t) , U the point (u, u) , and P the point $(t + a \cos t, t + a \sin t) = (u + a \cos u, u + a \sin u)$.

Since $\overline{PT} = \overline{PU} = a$, the triangle PTU is isosceles. Therefore its base angles, $\alpha = \angle PTU$ and $\beta = \angle PUT$ are equal. Since $\alpha = t - \frac{\pi}{4}$ and $\beta = 2\pi - \frac{3\pi}{4} - u = \frac{5\pi}{4} - u$, the relation $\alpha = \beta$ implies that

$$u + t = \frac{3\pi}{2} \quad (1).$$

Since $\overline{TU} = \text{distance}((t, t), (u, u)) = \sqrt{2(u-t)^2} = \sqrt{2}(u-t)$, we see that

$$\cos \alpha = \frac{\frac{1}{2}\overline{TU}}{\overline{PT}} = \frac{(u-t)/\sqrt{2}}{a}, \text{ so } u-t = \sqrt{2}a \cos \alpha, \text{ that is,}$$

$$u-t = \sqrt{2}a \cos\left(t - \frac{\pi}{4}\right) \quad (2). \text{ Now } \cos\left(t - \frac{\pi}{4}\right) = \sin\left[\frac{\pi}{2} - \left(t - \frac{\pi}{4}\right)\right] = \sin\left(\frac{3\pi}{4} - t\right),$$

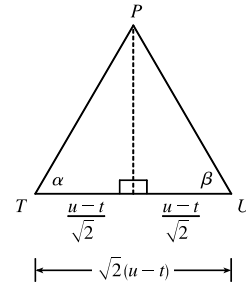
so we can rewrite (2) as $u-t = \sqrt{2}a \sin\left(\frac{3\pi}{4} - t\right)$ (2'). Subtracting (2') from (1) and

dividing by 2, we obtain $t = \frac{3\pi}{4} - \frac{\sqrt{2}}{2}a \sin\left(\frac{3\pi}{4} - t\right)$, or $\frac{3\pi}{4} - t = \frac{a}{\sqrt{2}} \sin\left(\frac{3\pi}{4} - t\right)$ (3).

Since $a > 0$ and $t < u$, it follows from (2') that $\sin\left(\frac{3\pi}{4} - t\right) > 0$. Thus from (3) we see that $t < \frac{3\pi}{4}$. [We have implicitly assumed that $0 < t < \pi$ by the way we drew our diagram, but we lost no generality by doing so since replacing t by $t + 2\pi$ merely increases x and y by 2π . The curve's basic shape repeats every time we change t by 2π .] Solving for a in

(3), we get $a = \frac{\sqrt{2}\left(\frac{3\pi}{4} - t\right)}{\sin\left(\frac{3\pi}{4} - t\right)}$. Write $z = \frac{3\pi}{4} - t$. Then $a = \frac{\sqrt{2}z}{\sin z}$, where $z > 0$. Now $\sin z < z$ for $z > 0$, so $a > \sqrt{2}$.

[As $z \rightarrow 0^+$, that is, as $t \rightarrow \left(\frac{3\pi}{4}\right)^-$, $a \rightarrow \sqrt{2}$.]



62. Consider the curves $x = \sin t + \sin nt$, $y = \cos t + \cos nt$, where n is a positive integer. For $n = 1$, we get a circle of radius 2 centered at the origin. For $n > 1$, we get a curve lying on or inside that circle that traces out $n - 1$ loops as t ranges from 0 to 2π .

$$\begin{aligned} \text{Note: } x^2 + y^2 &= (\sin t + \sin nt)^2 + (\cos t + \cos nt)^2 \\ &= \sin^2 t + 2 \sin t \sin nt + \sin^2 nt + \cos^2 t + 2 \cos t \cos nt + \cos^2 nt \\ &= (\sin^2 t + \cos^2 t) + (\sin^2 nt + \cos^2 nt) + 2(\cos t \cos nt + \sin t \sin nt) \\ &= 1 + 1 + 2 \cos(t - nt) = 2 + 2 \cos((1 - n)t) \leq 4 = 2^2, \end{aligned}$$

with equality for $n = 1$. This shows that each curve lies on or inside the curve for $n = 1$, which is a circle of radius 2 centered at the origin.

